Political Districting

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1 Introduction

Political districting is the task of partitioning a state into geographic-based districts for election purposes. These districts may elect a single representative (*single-member district*) or multiple representatives (*multi-member district*). The *voting method* determines how ballots in each district are combined to select representatives (e.g., majority, plurality, and ranked choice voting [97, 108]).

In the United States, districts typically are single-member and use plurality voting ("most votes wins"). This includes all congressional districts (which elect individuals to the US House of Representatives) and most state legislative districts (which elect individuals to the respective state governments). For example, Oklahoma is divided into five congressional districts as depicted in Figure 1 (which send representatives to the US Capitol in Washington, DC), 48 state senate districts (which send state senators to the Oklahoma State Capitol in Oklahoma City, OK), and 101 state house districts (which send state representatives to the Oklahoma State Capitol). The number of congressional seats that each state receives is determined by their respective populations, a process called apportionment [6], while the number of legislative districts is set by state law.

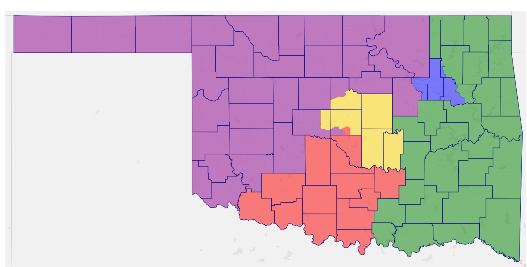

Figure 1: Oklahoma's five congressional districts, 2022-present

These districts are redrawn every ten years, following the census. A primary motivation is that district populations change over time and must be rebalanced.

Current practice in the USA is that most congressional districts satisfy 1-person deviation, e.g., each of Oklahoma's congressional districts was redrawn in 2022 to have a population of either 791,870 or 791,871 (according to 2020 census counts). Larger deviations of $\pm 5\%$ from the mean are permitted by the courts for legislative districts [32, 61]. These *population balance* constraints are not numerically specified in US federal law, but have emerged from the "one person, one vote" revolution of Supreme Court cases like *Baker v. Carr* (1962), *Wesberry v. Sanders* (1964), and *Reynolds v. Sims* (1964). Federal law in the USA also includes the Voting Rights Act, which requires that districts not dilute the voting power of minority groups [57], see *Thornburg v. Gingles* (1986) and *Allen v. Milligan* (2023). Simultaneously, the Equal Protection Clause of the 14th Amendment to the US Constitution prohibits race from "predominating" the districting process [61], see *Shaw v. Reno* (1993) and *Miller v. Johnson* (1995).

States impose additional laws on redistricting [32]. For example, most states require districts to be contiguous on the map. Other *traditional districting principles* include that districts should be compact and preserve political subdivisions (e.g., counties, cities, towns, wards), communities of interest, and cores of prior districts [82]. Despite these additional "soft" constraints, the set of feasible solutions remains astronomically large, enabling mapmakers to draw plans that favor a political party (*partisan gerrymander*) or incumbent politicians (*incumbent gerrymander*). Given that partisan-controlled state legislatures typically draw the lines, there is incentive to continue gerrymandering, and the Supreme Court has neglected to intervene, see *Rucho v. Common Cause* (2019).

Nevertheless, reform groups have had some success in establishing independent redistricting commissions (IRCs) to draw district lines. IRCs may be instructed to follow traditional redistricting principles, in what might be called procedural fairness [103]. Still, it has been observed that traditional redistricting principles may inadvertently favor one political party over another [26], in ways that are hard to predict [40]. This prompts the explicit consideration of other criteria like partisan fairness, competitiveness, or proportionality [15, 16, 30, 40, 52, 60, 96, 99].

For more information, we refer the reader to the recent book edited by Duchin and Walch [43] (an instant classic), the operations research survey of Ricca et al. [90], and the legal guides by Hebert et al. [61] and Davis et al. [32]. Other related surveys include [20, 55, 58, 59, 69, 75, 92, 105, 111].

Outline. Section 2 summarizes some of the criteria and objectives in districting, along with associated complexity results. Section 3 discusses sampling procedures which are used to understand the distribution of possible districting plans. Section 4 covers heuristic approaches to districting, including construction heuristics (which generate districts from scratch) and local search heuristics (which improve an initial plan with respect to given criteria). Section 5 provides mathematical optimization models for districting. We conclude in Section 6.

2 Criteria and Complexity

We can represent each state as a simple (undirected) graph $G = (V, E)$. The vertices V represent the basic geographic units (e.g., counties, census tracts, census blocks, voting precincts) used to construct districts, and the edges E indicate which pairs of geographic units are adjacent on the map. Figure 2 illustrates Oklahoma's county-level graph. In *rook adjacency*, geographic units u and v must share a border of positive length for the edge $\{u, v\}$ to be in E; the less common *queen adjacency* only requires u and v to meet at a point. Under rook adjacency, the graph G is planar and very sparse, with the numbers of vertices $n = |V|$ and edges $m = |E|$ satisfying $m \leq 3n - 6$ if $n \geq 3$; queen adjacency permits complete graphs with $m = \binom{n}{2}$ edges on "pizza pie" instances.

The vertices adjacent to vertex i constitute its neighborhood $N(i) = \{j \in V \mid \{i, j\} \in E\}$. The subgraph induced by a subset of vertices $S \subseteq V$ is denoted by $G[S] = (S, E(S))$ where $E(S)$ is the subset of edges with both endpoints in S. We say that a district $D \subseteq V$ is connected in the graph (or contiguous on the map) if its induced subgraph $G[S]$ is connected.

Each geographic unit $i \in V$ has an associated population p_i. Given a subset of geographic units $S \subseteq V$, their combined population is given by the shorthand $p(S) := \sum_{i \in S} p_i$. The ideal district population $p(V)/k$ is the state's total population $p(V)$ divided by the predefined number of districts k. The smallest and largest populations permitted in a district are denoted by L and U. In 1-person deviation, we have $L = \lfloor p(V)/k \rfloor$ and $U = \lceil p(V)/k \rceil$. In 10% deviation ($\pm 5\%$), we have $L = \lceil 0.95p(V)/k \rceil$ and $U = \lfloor 1.05p(V)/k \rfloor$.

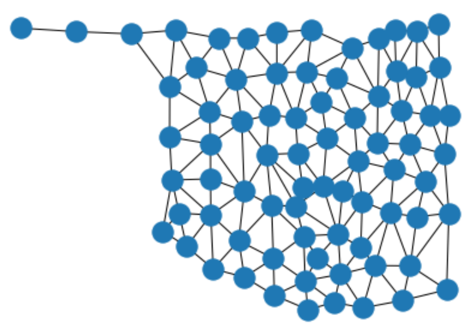

Figure 2: Oklahoma's county-level graph

The districting problem, in its simplest form, is to partition the vertices V into k districts $(D_1, D_2, \ldots, D_k)$ such that each district is contiguous and population-balanced, i.e., each district D_j should be connected and satisfy $L \leq p(D_j) \leq U$. Unfortunately, this problem is already NP-hard for unit populations ($p_i = 1$ for $i \in V$) and $L = U = 3$, see [44]. Even if the contiguity constraints are relaxed, NP-hardness persists as it can express the PARTITION problem [3].

In practice, the population balance constraints are less of an issue than the worst-case complexity would suggest. Intuitively, the reason is that hard instances of PARTITION involve large integers, while districts are often built from census blocks, which can have single digit (or zero) populations. Contiguity is often blamed for the practical difficulty, with Ricca et al. [90] stating that "it is particularly difficult to deal with and [is often] discarded from [districting] models and considered only *a posteriori*". We have also found the large size of districting instances to be a challenge, with many states having hundreds of thousands of census blocks. Another challenge is the many criteria that districting plans are supposed to satisfy and the many ways to quantify them.

As an example, consider compactness, which is the idea that a district should have a "nice" shape. Dozens of alternative compactness scores have been proposed over the years, with many viewing circles and squares to be the most compact [83]. In the optimization literature, the first was the moment-of-inertia (MOI), proposed by Hess et al. [64]. Taking inspiration from physics, the MOI of a single district $D \subseteq V$ is calculated as $\sum_{i \in D} p_i d_{ij}^2$ where d_{ij} is the Euclidean distance between the (centers of) geographic units i and j on the map and j is the district's "center", chosen so as to minimize this sum. A (once) popular score among political scientists is the length-width score, which compares the length and width of the district, with a ratio of one indicating ideal compactness. It is easy to draw districts that look awful but score well according this score [112]. The Polsby-Popper score [87], which is currently the most popular score in the redistricting literature and in expert testimony [37], is defined as $4\pi A/P^2$ where A is the district's area and P is the district's perimeter. It takes values between zero and one, with circular districts achieving a perfect score of one. This score, and others that rely on the district's perimeter length, are subject to the *Coastline Paradox*, which roughly states that district perimeters are not well-defined and depend on the choice of map projection and the "size of your ruler" [7, 8]. Recently, the number of *cut edges* between districts has become one of the more prominent compactness scores among mathematicians, computer scientists, statisticians, and operations researchers because it is simple, is less prone to abuse, reasonably agrees with the "eyeball test", and is well-suited for mathematical, computational, and probabilistic analysis [13, 34, 36, 65, 77, 88, 101].

Likewise, there are many ways to quantify how well political subdivisions, say, counties, are preserved. We could count how many *counties* are divided across districts, how many *times* counties are divided across districts, or examine the *extent* to which they are divided (e.g., a county that is divided 90/10 across two districts may be "less" split than a county that is divided 50/50). For a more involved discussion, we refer the reader to [12, 23, 54, 104]. Similar approaches can be used to quantify the preservation of communities of interest.

We refer the reader to other resources on quantifying or "operationalizing" partisan fairness, proportionality, and competitiveness [16, 30, 40, 52, 60, 96, 99].

3 Sampling Methods

Over the last twenty years, Massachusetts has had either 9 or 10 congressional districts, and all of them have elected Democrats. This is despite the fact that

30% to 40% of the state's votes went to Republicans. Does this indicate an intent to gerrymander? The answer turns out to be no. Duchin et al. [39] find that it is impossible to draw a Republican-majority district in Massachusetts built from voting precincts. The reason is that Republicans are distributed nearly uniformly throughout the state. Intuitively, if all precincts voted 40% Republican, then all districts built from precincts would likewise vote 40% Republican and thus elect Democrats. So, disproportionate outcomes are not necessarily a sign of intentional gerrymandering.

To arrive at intent, a better approach is to *randomly* draw districting plans to see what outcomes are likely to have occurred by chance. If a proposed or enacted plan is an outlier in this distribution of possible plans (say, with respect to the number of districts won by a particular party), then this may suggest an intent to gerrymander. To make this approach rigorous, we would need to determine which districting plans should not be sampled (e.g., if they violate population balance or contiguity) and what probability we should attach to each remaining plan (e.g., favoring compact plans over snaking, fractal-like districts). This is where *sampling*, *ensemble*, or "simulation" methods enter the picture.

Many ensemble methods take a Markov chain Monte Carlo (MCMC) approach [2, 28, 29, 34, 35, 47, 63, 114]. From an initial districting plan, the approach randomly "walks" to similar, neighboring plans, say, by flipping a vertex from one district to a neighboring district, or swapping two vertices between neighboring districts. These *flip* or *swap* neighborhoods have empirically been found to "mix" slowly, meaning that the approach may not adequately search or sample the solution space in a reasonable number of iterations, cf. [81].

This motivated DeFord et al. [36] to propose a new neighborhood called recombination. In ReCom, two adjacent districts are merged into a double district, a spanning tree is randomly drawn over their nodes, and an edge is deleted to split the tree into two subtrees which are taken as the two new districts (provided that they are feasible). Empirically, this approach gets "lost" quickly with, say, the 10,000th districting plan being nothing like the initial plan. Further, the closely associated *spanning tree distribution* favors compact districts [21, 88]. The GerryChain software package provides an open-source Python implementation [80], cf. a Julia implementation [91]. Despite their practical success, ReCom and other ensemble methods are not guaranteed to mix quickly [25].

Recent works seek to incorporate more "rules of the game" into the approach, like preserving political subdivisions [5, 30, 77] and ensuring minority representation [11, 22], cf. [27, 41, 42]. They may also use an explicit target distribution, like the spanning tree distribution [5, 21, 77], to make the approach more credible. In particular, McCartan and Imai [77] propose an entirely different approach called sequential Monte Carlo (SMC) which avoids the Markov Chain altogether; it builds a batch of districting plans by carving districts off one-by-one, also with random spanning trees and edge deletions. Kenny et al. [70] use the approach to evaluate the nationwide effects of partisan gerrymandering [78].

4 Heuristic Methods

Heuristics are inexact procedures used to find "good enough" solutions to an optimization problem [1]. In particular, construction heuristics build a feasible solution from scratch and have been proposed for political districting since at least 1961 [103]. We refer the reader to the survey of Becker and Solomon [13, Sec. 3.1] for a nice introduction and Ricca et al. [90, Sec. 3.1] for additional references to the literature. One high-level approach is to identify a set of k vertices to "seed" the districts and grow the districts outwards. Another high-level approach is to build districts one-at-a-time by carving them from the state.

Local search heuristics start with a feasible solution and (repeatedly) make small changes to improve its performance with respect to given criteria. Traditionally, local search heuristics for districting have used either a flip neighborhood or a swap neighborhood, sometimes with specialized techniques that exploit planarity to speed up the connectivity checks at each iteration [71, 72, 73]. To escape local optima, many researchers and practitioners adopt metaheuristic techniques such as simulated annealing, tabu search, and genetic algorithms. Again, we refer the reader to [13, 90] for pointers to many papers in this area.

The success of DeFord et al.'s recombination neighborhood [36] has prompted researchers to use it for heuristic optimization purposes. Some researchers apply ReCom in an MCMC fashion and pluck out those that are satisfactory or perform best. Duchin and Schoenbach [40] take this approach to find proportional districting plans. Cannon et al. [22] likewise take an unbiased ReCom walk for a small number of steps, but then restart the walk from the best-performing solution, with the aim of maximizing the number of minority-opportunity districts, cf. [11]. This "short bursts" approach has also been used in the search for plans that are Pareto optimal with respect to compactness and population balance [76]. Geodert et al. [56] use a mix of ReCom and flip moves, with flip moves becoming more prevalent throughout the procedure, to optimize weighted combinations of Black representation and compactness. Swamy et al. [98] also use a mix of ReCom and flip moves to draw districts for Arizona, considering compactness, partisan fairness, competitiveness, and the number of majority-minority districts. Other researchers find the *best* (rather than a random) move in the ReCom neighborhood, merging and redividing up to four districts at a time to minimize the number of county splits [93]. Because the ReCom neighborhood contains exponentially many districting plans—many more than the flip or swap neighborhoods—it is no longer feasible to use brute force enumeration to find the best plan in this neighborhood, thus requiring the solution of an optimization problem at each iteration.

Inspired by work of Henzinger et al. [62] on graph partitioning, Belotti et al. [14] propose another local search neighborhood for political districting called the h-hop neighborhood. The idea is that vertices deep within a district are required to stay in their current district, while vertices near a district border are permitted to switch to a different nearby district. More formally, a vertex $v \in D$ is permitted to switch to a different district D' (in the next iteration) if there is a vertex $v' \in D'$ with hop-based distance $\text{dist}_G(v, v') \leq h$. When the user-chosen parameter h is relatively small, say $h \in \{1, 2\}$, the associated optimization problem is relatively easy to solve. For example, Belotti et al. [14] apply the approach to optimize the Polsby-Popper compactness score.

5 Optimization Methods

This section covers mathematical optimization models for political districting. We categorize the models based on the their primary decision variables. Example Python codes for several districting models are available on GitHub at `https://github.com/AustinLBuchanan/Districting-Examples-2020`.

5.1 Using Hess Variables

Motivated by the landmark "one person, one vote" Supreme Court cases of the 1960s, Hess et al. [64, 107] proposed the first optimization model for political districting. The binary variable x_{ij} equals one when vertex $i \in V$ is assigned to the district centered at vertex $j \in V$ with $x_{jj} = 1$ indicating that j is a center.

$$\min \quad \sum_{i \in V} \sum_{j \in V} p_i d_{ij}^2 x_{ij} \tag{1a}$$

$$\text{s.t.} \quad \sum_{j \in V} x_{ij} = 1 \qquad \forall i \in V \tag{1b}$$

$$\sum_{j \in V} x_{jj} = k \tag{1c}$$

$$L x_{jj} \leq \sum_{i \in V} p_i x_{ij} \leq U x_{jj} \qquad \forall j \in V \tag{1d}$$

$$x_{ij} \leq x_{jj} \qquad \forall i, j \in V \tag{1e}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in V. \tag{1f}$$

The objective (1a) minimizes the moment-of-inertia. Assignment constraints (1b) ensure that each vertex is assigned to one district. Constraint (1c) requires k district centers to be selected. Population balance constraints (1d) require each district to have a population between L and U. The coupling constraints (1e) were not originally imposed by Hess et al. but help to strengthen the linear programming (LP) relaxation. As written, this model lacks contiguity constraints.

Due to computer and software limitations of the time, Hess et al. [64] solved this integer program heuristically. A main insight is that, after the district centers have been selected, the resulting subproblem is essentially a transportation problem, which can be solved efficiently, cf. [49, 53]. Afterwards, we have a partition of the vertices into districts $(D_1, D_2, \dots, D_k)$. Within each district D, identify the best vertex $j \in D$ to serve as its center, i.e., the vertex j that minimizes $\sum_{i \in D} p_i d_{ij}^2$. With these new district centers, re-solve the transportation problem and repeat until convergence, much like the popular k-means heuristic. See Lawless and Günlük [74] for a recent variant with representation constraints.

To impose contiguity, Shirabe [94, 95] proposes a formulation in which "flow" originates at each district center and is sent across the district's edges to "fuel" its other nodes, cf. [84, 102]. In particular, introduce the variable f^j_{uv} to indicate how much flow, originating from district center j, is sent across the directed edge (u, v). To the original n^2 Hess variables, this adds $2nm$ flow variables, which is $O(n^2)$ if G has $m = O(n)$ edges (true for planar G). The constraints below ensure that each non-center vertex is fueled (2a), flow of type j can enter vertex i only if i is assigned to j (2b), and flow cannot re-enter a center (2c).

$$\sum_{u \in N(i)} (f^j_{ui} - f^j_{iu}) = x_{ij} \qquad \forall i \in V \setminus \{j\},\ \forall j \in V \tag{2a}$$

$$\sum_{u \in N(i)} f^j_{ui} \leq (n-1)x_{ij} \qquad \forall i \in V \setminus \{j\},\ \forall j \in V \tag{2b}$$

$$\sum_{u \in N(j)} f^j_{uj} = 0 \qquad \forall j \in V \tag{2c}$$

$$f^v_{ij},\ f^v_{ji} \geq 0 \qquad \forall \{i,j\} \in E,\ \forall v \in V. \tag{2d}$$

These constraints are arguably the most popular contiguity constraints in the literature, e.g., being used by Validi et al [102] for county-level instances of the moment-of-inertia objective and Swamy et al. [99] for objectives relating to partisan symmetry, efficiency gap, and competitiveness. Shahmizad and Buchanan [93] use them to solve the so-called *county clustering problem* which provides a strong lower bound on the minimum number of county splits, cf. [23].

Another approach for imposing contiguity uses separator inequalities [19, 24, 48, 84, 106], cf. [89]. When applied to Hess variables, they take the form

$$(a, b\text{-separator inequality}) \qquad x_{aj} + x_{bj} \leq 1 + \sum_{c \in C} x_{cj}$$

where a and b are nonadjacent vertices and $C \subseteq V \setminus \{a, b\}$ is an a, b-separator, i.e., there is no path between a and b in the subgraph $G[V \setminus C]$. Because there are exponentially many of these constraints, they are typically applied in a branch-and-cut fashion. Validi et al. [102] show that, for planar graphs, the associated separation problem is solvable in time $O(n^2 \log n)$ and $O(n^2)$ for fractional and integer x, respectively. In their experiments with the MOI objective, they find that few of these inequalities are necessary, allowing for some instances with up to $1,500$ vertices to be solved exactly. By exploiting the population balance constraints, they also write stronger length-U a, b-separator inequalities.

Finally, we consider some contiguity constraints that are fast in practice, but are invalid in the sense that they cut off feasible points. First, we have the *tree-based contiguity constraints* of Zoltners and Sinha [115]; after finding a shortest paths tree rooted at a particular vertex j, impose constraints of the form $x_{ij} \leq x_{vj}$ for $i \in V \setminus \{j\}$ where v is the predecessor of i in the tree. These constraints can been relaxed to *distance-based contiguity constraints* [31, 60, 79, 85], which allow more solutions and take the form $x_{ij} \leq \sum_v x_{vj}$, where the sum is over all neighbors $v \in N(i)$ that are nearer to j, i.e., $\text{dist}(v, j) < \text{dist}(i, j)$. Önal and Patrick [85] use these constraints to draw plans for Illinois that fare better than the enacted plan with respect to county splits and minority representation. To allow even more solutions, we can use *DAG-based contiguity constraints* [93] in which the edges of G are oriented away from j in an acyclic fashion and impose $x_{ij} \le \sum_v x_{vj}$, where the sum is over all in-neighbors of i in the orientation.

In the spirit of Beasley [10], Hojati [66] proposes a Lagrangian relaxation of the Hess model in which population balance (1d) is first imposed strictly ($L = U$). Then, the population balance constraints and assignment constraints (1b) are relaxed, with their violation penalized in the objective function with respect to the Lagrange multipliers. This results in a Lagrangian relaxation model that is easy to solve combinatorially and that satisfies the "integrality property", implying that the Lagrangian relaxation bound coming from the best Lagrange multipliers equals that of the LP relaxation. Motivation for using Lagrangian relaxation instead of LP relaxation included that the Lagrangian was easier to solve and less memory-intensive, especially using LP software of the day. Later, Validi et al. [102] propose a similar Lagrangian relaxation model where $L < U$, and use it to fix $x_{ij} = 0$ when it can be deduced that $x_{ij} = 1$ would be suboptimal, making rigorous earlier approaches that heuristically fix such variables [85]. They also exploit the contiguity constraints in the Lagrangian relaxation, using it to solve instances of the Hess model with $1,500$ vertices.

5.2 Using Labeling Variables

Another class of integer programming models uses labeling or assignment variables of the form x_{ij} which equal one when vertex $i \in V$ is assigned to district $j \in [k] := \{1, 2, \ldots, k\}$. Because the number of districts is usually much smaller than the number of vertices, labeling models are typically smaller than Hess models. For example, if applied to Oklahoma's $|V| = 1,205$ census tracts and $k = 5$ congressional districts, there would be $6,025$ labeling variables or $1,452,025$ Hess variables. A typical use of labeling models is to minimize the number of cut edges between districts [13, 17, 46, 68, 93, 101]. So, introduce a binary variable y_e for each edge $e \in E$ indicating whether it is cut and write:

$$\min \quad \sum_{e \in E} y_e \tag{3a}$$

$$\text{s.t.} \quad x_{uj} - x_{vj} \le y_e \qquad \forall e = \{u, v\} \in E,\ \forall j \in [k] \tag{3b}$$

$$\sum_{j=1}^{k} x_{ij} = 1 \qquad \forall i \in V \tag{3c}$$

$$L \le \sum_{i \in V} p_i x_{ij} \le U \qquad \forall j \in [k] \tag{3d}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i \in V,\ \forall j \in [k] \tag{3e}$$

$$y_e \in \{0, 1\} \qquad \forall e \in E. \tag{3f}$$

One notable drawback of labeling formulations is symmetry; the same districting plan can be represented in $k!$ different ways by permuting the district labels [68]. Consequently, even if we added valid inequalities to recover the convex hull in the x-space of variables, the LP relaxation would still allow the point $(\bar{x}, \bar{y})$ where $\bar{y} = \mathbf{0}$ by setting $\bar{x}_{ij} = 1/k$ for all $i \in V$ and $j \in [k]$, see [101]. One way to avoid this symmetry is to apply the extended formulation for partitioning orbitopes due to Faenza and Kaibel [45], cf. [101], which forces the districts to be sorted lexicographically. Another computational speedup comes by exploiting the population balance constraints, specifically the population lower bounds L, in a procedure called L-fixing [101]. Many forms of the contiguity constraints that were previously discussed for Hess models can be adapted to the labeling context, such as the Shirabe model [94, 95, 101], separator constraints [101], as well as a notable single-commodity flow formulation from Hojny et al. [67], cf. [18]. Ferreira et al. [46] propose valid inequalities and an extended formulation to strengthen the cut edge LP relaxation which are also very helpful [101].

Labeling models can also be applied to minimize the sum of district perimeters [101], as this amounts to a *weighted* cut edges objective. Belotti et al. [14] extend the model to handle the Polsby-Popper score in a mixed-integer second-order cone program (MISOCP). They apply the MISOCPs to draw compact majority-minority districts, including a case study motivated by the Supreme Court case *Allen v. Milligan* (2023) in which they draw a compact plan for Alabama that has two Black-majority districts. Fravel et al. [50] extend the labeling model to handle or estimate nonconvex objectives relating to the Polsby-Popper score, Black representation, and partisan outcomes. Arredondo et al. [4] use a labeling model focused on Indigenous representation in Mexico.

5.3 Using District Variables

In a completely different class of optimization models, introduce a binary variable x_D and a cost c_D for each possible district $D \in \mathcal{D}$. Then, a set partitioning model over the district variables can be written as follows.

$$\min \quad \sum_{D \in \mathcal{D}} c_D x_D \tag{4a}$$

$$\text{s.t.} \quad \sum_{D \in \mathcal{D} : i \in D} x_D = 1 \qquad \forall i \in V \tag{4b}$$

$$\sum_{D \in \mathcal{D}} x_D = k \tag{4c}$$

$$x_D \in \{0, 1\} \qquad \forall D \in \mathcal{D}. \tag{4d}$$

When Garfinkel and Nemhauser [51] first introduced this model for districting, they took a two-step approach. The first step was to enumerate all suitable districts $D \in \mathcal{D}$, and the second step was to solve the resulting set partitioning model (4). Nowadays, a typical strategy for "solving" models like these (with exponentially many variables) is to first solve the LP relaxation using column generation and then solve the associated integer program over the generated columns. A related optimization-based heuristic was adopted by Mehrotra et al. [79]. More recently, Gurnee and Shmoys [60] take a column generation approach, using stochastic hierarchical partitioning to quickly generate many columns, with an eye towards fairness. To truly *solve* the set partitioning IP, one would likely need to take a branch-and-price approach [9], which is essentially branch-and-bound where the LP relaxations are solved using column generation. However, to the best of our knowledge, the political districting literature does not contain any *exact* branch-and-price implementations, although Borndörfer et al. [18] test their approach on related commercial territory design instances.

5.4 Using Spanning Tree Edge Variables

Recognizing that connected districts admit spanning trees, we could define a variable x_e that equals one if the endpoints of edge e belong to the same district and if this edge is selected as part of the district's spanning tree. By adding $k-1$ other edges to these spanning trees' edges, we can obtain a spanning tree for the entire graph. Indeed, there is a linear-size extended formulation for spanning trees in planar graphs due to Williams [109, 110]; see [86, 100] for corrections to this model. With this modeling primitive, we can write a *linear-size* formulation for partitioning the vertices of a planar graph into k components *that is integral.* Unfortunately, Zhang et al. [113] find that, when population balance is imposed, the integrality of the formulation is destroyed, and it performs worse than the Hess model. It is an open question whether this spanning tree model can be redeemed with alternative population balance constraints.

6 Conclusion

Political districting remains a challenging problem for optimization methods. This is partially due to the large size of districting instances and the many objectives and criteria that one must deal with. This is not to say that optimization cannot have an impact, but rather to emphasize the mathematical and computational ingenuity that is required, as well as the familiarity with the entirety of the districting literature, including political science, computer science, mathematics, and litigation [38]. Optimization *can* be a powerful tool for districting, to illuminate tradeoffs between districting criteria and to show the limits of what is possible. Indeed, in an amicus brief cited by the Supreme Court in *Allen v. Milligan* (2023), a team of comptutational redistricting experts wrote that "optimization algorithms are well-suited to the task of generating [remedial plans in VRA litigation]... as they can identify innovative combinations of geography that better comply with multiple traditional redistricting principles than any individual mapmaker is likely to find manually through trial and error" [33].

See Also

Graph Partitioning
Optimal Transport in Location-Allocation Problems

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