

A Note on “A linear-size zero-one programming model for the minimum spanning tree problem in planar graphs”

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Abstract

In the paper “A linear-size zero-one programming model for the minimum spanning tree problem in planar graphs” (Networks 39(1):53–60, 2002), Williams introduced an extended formulation for the spanning tree polytope of a planar graph. This formulation is remarkably small (using only $O(n)$ variables and constraints) and remarkably strong (defining an integral polytope). In this note, we point out that Williams’ formulation, as originally stated, is incorrect. Specifically, we construct a binary feasible solution to Williams’ formulation that does not represent a spanning tree. Fortunately, there is a simple fix, which is to restrict the choice of the root vertices in the primal and dual spanning trees, whereas Williams explicitly allowed them to be chosen arbitrarily. The same flaw and fix apply to a subsequent formulation of Williams (“A Zero-One Programming Model for Contiguous Land Acquisition.” Geographical Analysis 34(4): 330-349, 2002).

Keywords: spanning tree; planar graph; planar dual; extended formulation; extension complexity; spanning arborescence; integer programming;

1 Introduction

In 2002, Williams [13] proposed an extended formulation for spanning trees of a planar graph. The formulation is remarkably small (using only linearly many variables and constraints) and remarkably strong (defining an integral polytope). This shows that the spanning tree polytope of a simple planar graph $G = (V, E)$ has extension complexity $O(n)$, where $n = |V|$ is the number of its vertices.

This result is frequently mentioned as an exemplary example of the power of extended formulations [2, 5, 7], and is a key ingredient in several subsequent extended formulations [3, 6, 8]. Indeed, a recent paper by Fiorini et al. [6], which gives size $O(g^{1/2}n^{3/2} + g^{3/2}n^{1/2})$ extended formulations for spanning trees in graphs of genus g , depends crucially on Williams’ result. The spanning tree formulation and a subsequent formulation for vertex-induced connectivity by Williams [14] are also used computationally [1, 9, 12, 15].

In this note, we show that Williams’ spanning tree formulation is incorrect as stated. Specifically, we construct a binary feasible solution to Williams’ spanning tree formulation that does not represent a spanning tree. Fortunately, a small tweak corrects the formulation. The change is to restrict the choice of the root vertices in the primal and dual spanning trees, whereas Williams explicitly allowed them to be chosen arbitrarily. That this restriction on the roots results in a correct formulation was first observed by Pashkovich and Kaibel (and appears in Pashkovich’s dissertation [10]), but the result and its proof have not appeared outside of this dissertation, nor does the dissertation point out the error with Williams’ original formulation. Additionally, we prove the converse statement: if their Root Rule is not followed, then Williams’ formulation will (always) allow a solution that is not a spanning tree. This shows that the Root Rule is, in some sense, *the* way to fix Williams’ formulation.

Then, we give a counterexample to a subsequent formulation from Williams for vertex-induced connected subgraphs of planar graphs, which was introduced in the context of acquiring contiguous parcels of land [14]. That this second formulation is incorrect as stated is perhaps unsurprising given that it relies on the correctness of the spanning tree formulation. Fortunately, the same Root Rule patches it.

1.1 Terminology and Notation

Williams’ formulation uses the notion of the planar dual graph (defined below), which may have parallel edges and loops even when the graph coming from the original application does not. So, in an effort to keep the exposition uniform and general, all graphs considered here will be *multigraphs*, which are permitted to have parallel edges and loops. In this case, edges e are not uniquely identified by their endpoints u and v , so we will not write $e = \{u, v\}$, contra Williams [13] and Pashkovich [10].

In this note, $G = (V, E)$ will be a connected, undirected multigraph with vertex set V and edge set E . We call G the *primal* graph, or simply the “primal.” Replacing each undirected edge of G by its directed counterpart(s) yields the *bidirected* primal graph $\vec{\vec{G}} = (V, \vec{\vec{E}})$, or simply the “bidirected primal.” When an edge is a loop in G , we intend for $\vec{\vec{G}}$ to have only one associated directed edge. Undirected edges in G that have distinct endpoints result in two oppositely directed edges in

$\vec{\vec{G}}$.

We assume that G is embedded in the plane so that no two of its edges cross (i.e., a planar embedding). In this case, there is an associated graph $G^* = (V^*, E^*)$ called the *planar dual* of G , or simply the “dual.” This dual graph G^* is created as follows¹. Place one dual vertex inside each face of the embedding of G , including the exterior face. Place a dual edge across each primal edge, and connect it to the dual vertices representing the faces on either side of the primal edge. These two faces are identical when the primal edge is a bridge, yielding a loop in G^* . Figure 1 gives an example primal/dual pair of graphs.

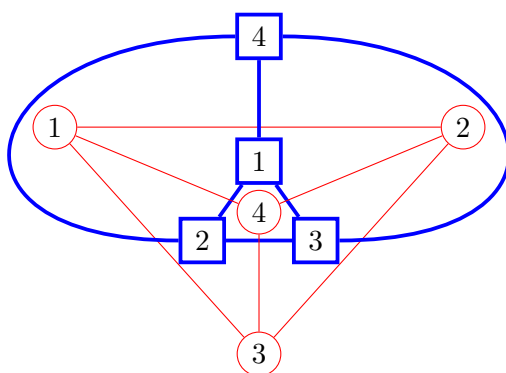


Figure 1: A planar embedding of “primal” G with round vertices, and its planar “dual” G^* with square vertices.

Note that G^* may have parallel edges and loops even when G does not. For example, this is true if G is a cycle with k edges, in which case G^* consists of two vertices with k parallel edges between them. Also, if G is a tree with k edges, then G^* consists of a single vertex with k loops.

Replacing each undirected edge of G^* by its directed counterpart(s) gives the bidirected dual graph $\vec{\vec{G}}^* = (V^*, \vec{\vec{E}}^*)$, or simply the “bidirected dual.” Figure 2 summarizes the notation.

As is standard, $\delta_H(i)$ denotes the subset of edges incident to vertex i in an undirected graph H , and $\delta_H^-(i)$ denotes the subset of directed edges entering vertex i in a directed graph H . The function q_H maps an undirected edge to the set of its endpoints in graph H , as in $q_H(e) = \{u, v\}$. Note that $q_H(e)$ will be a singleton when e is a loop.

There is a natural bijection between the edges of the primal and dual, based on their crossings. Following Diestel [4], the notation e will often refer to a primal edge, and e^* to its associated dual edge from the bijection.

¹The notation G^* for the dual is common and is used, for example, by Tutte [11] and Diestel [4].

	Undirected		Directed
Primal	G	$\xrightarrow{\text{bidirect}}$	\overleftrightarrow{G}
	\updownarrow		
Dual	G^*	$\xrightarrow{\text{bidirect}}$	$\overleftrightarrow{G^*}$

Figure 2: The names of the multigraphs in this note.

2 Spanning Tree Formulation

Williams' formulation [13] exploits the complementary nature of spanning trees in the primal and dual. For example, consider the primal spanning tree in Figure 3 indicated by the solid lines. The spanning tree drawn in the dual is the unique spanning tree that does not cross the edges of the primal spanning tree. Observe that exactly one edge is chosen from each pair of crossing edges. For example, the edge connecting primal vertices 1 and 4 is chosen but not the crossing edge that connects dual vertices 1 and 2. We will formalize these relationships in Lemma 2.

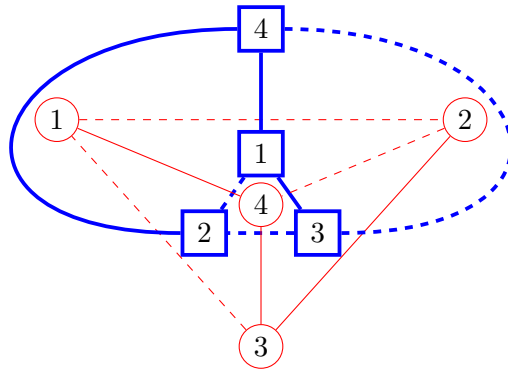


Figure 3: The complementary nature of primal and dual spanning trees. Solid edges are chosen and dashed edges are not.

Williams' formulation is as follows². It models a spanning arborescence, which is a directed version of a spanning tree in which all edges are pointed away from a root vertex, in which case each vertex (besides the root) has one incoming edge. For each primal edge e and for each of its endpoints i , there is a binary variable $x_{e,i}$ representing the choice to select edge e and orient it towards vertex i . There are similar binary variables $y_{e^*,u}$ for dual edges e^* and endpoints u . Recall that e^* is the notation for the dual edge that crosses primal edge e . Williams states that a primal root $r \in V$ and dual root $r^* \in V^*$ are to be picked arbitrarily.

$$\sum_{e \in \delta_G(i)} x_{e,i} = 1 \quad \forall i \in V \setminus \{r\} \quad (1)$$

$$\sum_{e \in \delta_G(i)} x_{e,i} = 0 \quad i = r \quad (2)$$

$$\sum_{e^* \in \delta_{G^*}(u)} y_{e^*,u} = 1 \quad \forall u \in V^* \setminus \{r^*\} \quad (3)$$

$$\sum_{e^* \in \delta_{G^*}(u)} y_{e^*,u} = 0 \quad u = r^* \quad (4)$$

$$\sum_{i \in q_G(e)} x_{e,i} + \sum_{u \in q_{G^*}(e^*)} y_{e^*,u} = 1 \quad \forall e \in E \quad (5)$$

$$x_{e,i} \in \{0, 1\} \quad \forall e \in \delta_G(i), i \in V \quad (6)$$

$$y_{e^*,u} \in \{0, 1\} \quad \forall e^* \in \delta_{G^*}(u), u \in V^*. \quad (7)$$

The indegree constraints (1), (2), (3), (4) ensure that the roots r and r^* have zero incoming edges and that all other vertices have one incoming edge. The crossing edge constraints (5) typically take the form $x_{e,i} + x_{e,j} + y_{e^*,u} + y_{e^*,v} = 1$, where $\{i, j\}$ and $\{u, v\}$ are the endpoints of e and e^* , respectively. One exception is when e is a loop, in which case the constraint will take the form $x_{e,i} + y_{e^*,u} + y_{e^*,v} = 1$. The other exception is when e is a bridge, meaning e^* will be a loop, and the constraint will take the form $x_{e,i} + x_{e,j} + y_{e^*,u} = 1$. However, in any feasible solution these loop variables will take a value of zero since the corresponding bridge variables will sum to one, so the loop variables can be removed from the formulation. If desired, the variables in constraints (2) and (4) can also be removed.

²We make minor modifications. First, we model a spanning arborescence (instead of a spanning anti-arborescence) because arborescences are perhaps more common and also because this follows Pashkovich's dissertation. Second, Williams removes all edges pointing away from each terminus because the associated variables would end up being zero anyways in an anti-arborescence. Instead, like in Pashkovich's dissertation, we retain the edges pointing towards the roots. Third, Williams uses a variable x_{ij} for edge $e = (i, j)$, but due to the possibility for parallel edges, we instead use variables $x_{e,j}$, like in Pashkovich's dissertation, requiring us to refer to edge subsets $\delta(\cdot)$ instead of vertex neighborhoods. Fourth, we introduce the notation $q(\cdot)$ to correctly handle loops in the constraints (5).

As given, the formulation has $4|E| - b - s$ variables, where b and s are the number of bridges and loops, respectively, in G . If $G = (V, E)$ is planar and *simple*, the number of variables is $O(n)$, where $n = |V|$. This is because simple planar graphs $G = (V, E)$ with at least three vertices satisfy $|E| \leq 3|V| - 6$, which holds by Euler's polyhedral formula.

Spanning tree counterexample. Figure 4 gives a counterexample to Williams' formulation [13]. Here, we have chosen $r = 4$ and $r^* = 4$ as the primal and dual roots, respectively. Observe that each primal and dual vertex (besides the roots) has precisely one incoming solid edge, so the indegree constraints (1) and (3) are satisfied. And, the roots have zero incoming solid edges, so constraints (2) and (4) are satisfied. Finally, the crossing edge constraints (5) are satisfied. Yet, the selected primal edges form a directed cycle on vertices 1, 2, and 3; they do not form a spanning arborescence.

Later in this note we give a general procedure for constructing such counterexamples when they exist; see the proof of Lemma 3 for the details.

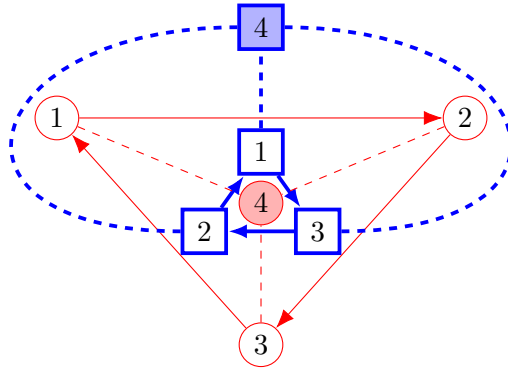


Figure 4: A counterexample to the spanning tree formulation of Williams [13].

2.1 Spanning Tree Fix

To fix Williams' formulation, we can use the following Root Rule [10].

Root Rule (Due to Pashkovich and Kaibel). *Arbitrarily pick a face $r^* \in V^*$ as the dual root. Then, pick a vertex $r \in V$ from that same face to be the primal root.*

Figure 5 gives examples of good and bad choices for the primal and dual roots.

Lemma 1 (folklore). *Let $G = (V, E)$ be a directed graph. The edge subset $\hat{E} \subseteq E$ is a spanning arborescence rooted at vertex $r \in V$ if and only if the following hold:*

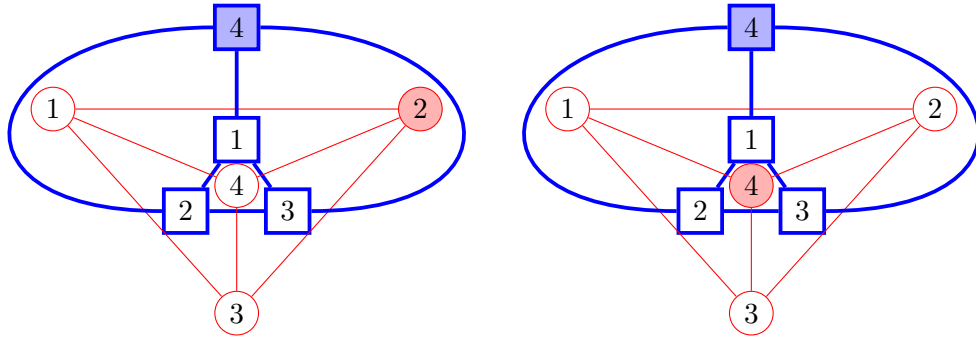


Figure 5: A good choice (left) and a bad choice (right) for the roots.

1. for each non-root $i \in V \setminus \{r\}$, exactly one directed edge incoming to i belongs to \hat{E} , i.e., $|\delta_G^-(u) \cap \hat{E}| = 1$;
2. no edge incoming to the root r belongs to \hat{E} , i.e., $|\delta_G^-(r) \cap \hat{E}| = 0$;
3. the subgraph (V, \hat{E}) contains no directed cycles.

Lemma 2 (see Theorem XI.6 of Tutte [11]). *Suppose G is connected. The edge subset $T \subseteq E$ is the edge set of a spanning tree of G if and only if $T^* := \{e^* \mid e \in E \setminus T\}$ is the edge set of a spanning tree of G^* .*

With these two lemmata, it can be shown that the Root Rule fixes the formulation, which we prove for completeness. The idea behind the proof originates with Pashkovich and Kaibel (see Lemma 3.4 in [10]).

Theorem 1 (Pashkovich and Kaibel). *If the Root Rule is followed, then Williams' spanning tree formulation is correct.*

Proof. Suppose the Root Rule is followed. We are to show that a binary vector \hat{x} represents a spanning arborescence of \vec{G} rooted at r if and only if there exists a binary vector \hat{y} such that (\hat{x}, \hat{y}) satisfies the formulation.

(\implies) Suppose that \hat{x} represents a spanning arborescence of \vec{G} . Undirecting these edges gives a spanning tree of G with edge set T . By Lemma 2, $T^* := \{e^* \mid e \in E \setminus T\}$ is the edge set of a spanning tree of G^* . Directing the edges of T^* away from r^* gives a spanning arborescence of \vec{G}^* rooted at r^* . Let \hat{y} be its characteristic vector. Since \hat{x} and \hat{y} represent spanning arborescences rooted at r and r^* , respectively, and by Lemma 1, (\hat{x}, \hat{y}) satisfies the indegree constraints and binary restrictions from Williams' formulation. Finally, (\hat{x}, \hat{y}) satisfy constraints (5) since they represent orientations of the complementary spanning trees T and T^* .

(\Leftarrow) Suppose that (\hat{x}, \hat{y}) satisfies the formulation. We show that \hat{x} represents a spanning arborescence of \overleftrightarrow{G} by showing it satisfies the properties from Lemma 1. Since \hat{x} satisfies the indegree constraints (1) and (2), properties 1 and 2 of Lemma 1 are satisfied. So, it suffices to show that \hat{x} satisfies property 3, which we show via Pashkovich and Kaibel’s infinite nested cycles.

For contradiction purposes, suppose \hat{x} selects the edge set \hat{C}_1 of a directed cycle (and possibly other edges too). The cycle does not contain the primal root r by constraint (2). It creates two distinct regions in the planar embedding: the interior and the exterior of the cycle. Let R_1 be the region that does *not* contain the primal root r . By the Root Rule, the dual root r^* also does not belong to R_1 . Now, the region R_1 contains at least one dual vertex, say $u \in V^* \setminus \{r^*\}$. By the indegree constraints (3), there is one dual edge e^* that is selected and oriented towards u , and by the crossing edge constraints (5) this edge cannot cross the cycle \hat{C}_1 into R_1 . Thus, the other endpoint of e^* , say v , also belongs to R_1 . Repeatedly tracing the directed edges backwards in this way shows that \hat{y} selects at least one directed cycle \hat{C}_1^* of \overleftrightarrow{G}^* within R_1 . This dual cycle \hat{C}_1^* also creates two regions: its interior and exterior. Let the region that does not contain r (and thus does not contain r^*) be called R_1^* . By the same arguments as before, \hat{x} selects a directed cycle of G that lies within R_1^* . Repeating this procedure gives an infinite sequence of nested primal/dual cycles $\hat{C}_1, \hat{C}_1^*, \hat{C}_2, \hat{C}_2^*, \dots$ that are edge disjoint. This contradicts that \overleftrightarrow{G} and \overleftrightarrow{G}^* have finitely many edges, implying that \hat{x} satisfies property 3. So, the directed edges selected by \hat{x} satisfy all properties from Lemma 1 and thus form a spanning arborescence of \overleftrightarrow{G} . \square

Observe that any choice of primal and dual roots satisfies the Root Rule when G is a tree or a cycle. First, planar embeddings of trees have only one face, meaning that there is only one choice for the dual root r^* , and all primal vertices lie on this face. Second, in the case of a cycle, there are two faces: interior and exterior. Again, all primal vertices lie on these faces, so either choice for the dual root will end up satisfying the Root Rule.

Remark 1. *If G is a tree or a cycle, then any choice of primal and dual roots satisfies the Root Rule.*

Lemma 3. *If G is neither a tree nor a cycle, then there is a choice of primal and dual roots such that Williams’ spanning tree formulation is incorrect.*

Proof. Suppose that G is neither a tree nor a cycle. We select primal and dual roots r and r^* and construct a feasible solution (\hat{x}, \hat{y}) to the formulation such that \hat{x} does not represent a spanning arborescence rooted at r .

Because G is not a tree (and by assumption throughout this note that G is connected), it contains an undirected cycle. Direct its edges into a directed cycle

$\hat{C} \subset \overset{\leftrightarrow}{E}$. Create a breadth-first search (BFS) tree in G emanating from this cycle and let its directed edges be $\hat{T} \subset \overset{\leftrightarrow}{E}$. Let $r \in V$ be a leaf vertex in this BFS tree and let the directed edge pointing to r be $e_r \in \hat{T}$. (That r and e_r exist outside of this cycle holds because G is connected but is itself not a cycle.) Let \hat{x} be the characteristic vector of $\hat{C} \cup \hat{T} \setminus \{e_r\}$. It can be observed that \hat{x} satisfies constraints (1) and (2).

By undirecting the edges of $\hat{C} \cup \hat{T} \setminus \{e_r\}$ we obtain the edge subset $\hat{E} \subset E$. Let $\hat{E}^* := \{e^* \mid e \in E \setminus \hat{E}\}$. We argue that \hat{E}^* induces a subgraph of G^* with two components: a tree and a 1-tree (i.e., a tree that has one extra edge). To see this, let e_c be an edge from the directed primal cycle \hat{C} . Let S be the spanning tree of G obtained by undirecting the edges of $(\hat{C} \setminus \{e_c\}) \cup \hat{T}$. Then, by Lemma 2, the dual edge subset $S^* := \{e^* \mid e \in E \setminus S\}$ is a spanning tree of G^* . Then, $S^* \setminus \{e_c^*\}$ is a forest containing two trees and $\hat{E}^* = (S^* \setminus \{e_c^*\}) \cup \{e_r^*\}$ induces a tree and a 1-tree.

Direct the edges of the 1-tree's cycle into a directed cycle \hat{C}^* , and let \hat{T}^* be the directed edges of a BFS tree in (V^*, \hat{E}^*) emanating from the cycle. Pick a vertex r^* from the tree component of (V^*, \hat{E}^*) and direct the edges of the tree component away from r^* , giving the subset \hat{T}_t^* of directed edges. Let \hat{y} be the characteristic vector of $\hat{C}^* \cup \hat{T}^* \cup \hat{T}_t^*$. It can be observed that \hat{y} satisfies constraints (3) and (4). Finally, constraints (5) are satisfied because \hat{x} and \hat{y} are orientations of the complementary edge subsets \hat{E} and \hat{E}^* .

So, (\hat{x}, \hat{y}) satisfies all constraints of the formulation, but \hat{x} does not represent a spanning arborescence of $\overset{\leftrightarrow}{G}$. Therefore, the formulation is incorrect under this choice of primal root r and dual root r^* . \square

By Theorem 1, Remark 1, and Lemma 3, we have the following theorem.

Theorem 2. *Williams' spanning tree formulation is correct if and only if the Root Rule is followed.*

Williams [13] notes that his formulation has a totally unimodular constraint matrix, implying that his formulation is integral. Pashkovich [10] observes in a footnote that total unimodularity persists when G and G^* are multigraphs, as long as each loop in the undirected graphs results in one directed loop in the bidirected graphs. Since we are primarily concerned with the formulation's correctness, and not its integrality, we refer the reader to page 42 of Pashkovich [10] for the proof.

3 Connected Subgraph Formulation

In a second paper [14], Williams adapts his spanning tree formulation so that it selects contiguous parcels of land. In the graph context, a feasible solution is a subset $S \subseteq V$ of vertices that induces a connected subgraph $G[S]$. Again, he

exploits planar graph duality, but the task of selecting only a *subset* of the *vertices* is different and uses different variables.

For each primal vertex $i \in V$, there is a variable u_i representing the decision whether or not to include it in S . There are two sets of primal edge variables, $x_{e,i}$ and $y_{e,i}$, that together select a spanning arborescence of \vec{G} . The variables $x_{e,i}$ select an arborescence that spans *the vertices selected by the u_i variables* (the “sub-arborescence”), and the variables $y_{e,i}$ represent the other edges of the spanning arborescence of \vec{G} . The variables $z_{e^*,u}$ represent a spanning arborescence of G^* . As before, $e^* \in E^*$ is the dual edge that crosses $e \in E$.

The formulation is as follows, where Williams again states that the primal and dual roots can be chosen arbitrarily. The first five constraints are identical to the spanning tree formulation, except that the selected primal edges are broken into two parts: $x_{e,i}$ and $y_{e,i}$. The last few constraints relate the vertex and edge decisions for the sub-arborescence³.

$$\sum_{e \in \delta(i)} (x_{e,i} + y_{e,i}) = 1 \quad \forall i \in V \setminus \{r\} \quad (8)$$

$$\sum_{e \in \delta(i)} (x_{e,i} + y_{e,i}) = 0 \quad i = r \quad (9)$$

$$\sum_{e^* \in \delta(u)} z_{e^*,u} = 1 \quad \forall u \in V^* \setminus \{r^*\} \quad (10)$$

$$\sum_{e^* \in \delta(u)} z_{e^*,u} = 0 \quad u = r^* \quad (11)$$

$$\sum_{i \in q_G(e)} x_{e,i} + \sum_{i \in q_G(e)} y_{e,i} + \sum_{u \in q_{G^*}(e^*)} z_{e^*,u} = 1 \quad \forall e \in E \quad (12)$$

$$\sum_{j \in q_G(e)} x_{e,j} \leq u_i \quad \forall i \in q_G(e), e \in E \quad (13)$$

$$\sum_{e \in \delta(i)} x_{e,i} \leq u_i \quad \forall i \in V \setminus \{r\} \quad (14)$$

$$\sum_{i \in V} u_i - \sum_{i \in V} \sum_{e \in \delta(i)} x_{e,i} = 1 \quad (15)$$

$$x_{e,i} \in \{0, 1\} \quad \forall e \in \delta_G(i), i \in V \quad (16)$$

$$y_{e,i} \in \{0, 1\} \quad \forall e \in \delta_G(i), i \in V \quad (17)$$

$$z_{e^*,u} \in \{0, 1\} \quad \forall e^* \in \delta_{G^*}(u), u \in V^* \quad (18)$$

$$u_i \in \{0, 1\} \quad \forall i \in V. \quad (19)$$

³Originally, Williams imposed cardinality constraints on the number p of vertices (parcels) and the number $p - 1$ of edges. Instead, we write the more general constraint (15).

Connected subgraph counterexample. Figure 6 gives a planar embedding of a primal graph on 6 round vertices. The dual vertices are squares.

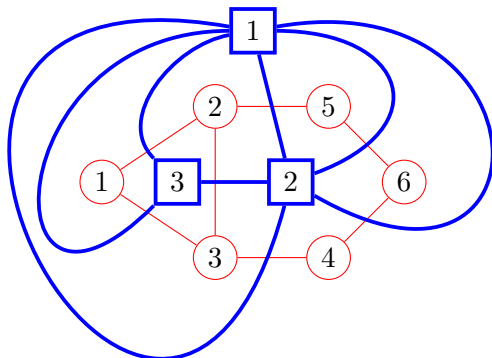


Figure 6: A planar embedding of a graph G and its dual G^* .

Figure 7 gives a counterexample to Williams' connected subgraph formulation [14]. Here, we have chosen $r = 6$ and $r^* = 3$ as the primal and dual roots, respectively. Observe that all constraints of the formulation above are satisfied, and yet the gray-filled primal vertices (representing vertices i with $u_i = 1$) induce a disconnected subgraph. The primal edges selected by the x variables (resp. y variables) are solid (resp. dashed). The dual edges selected by the z variables are solid.

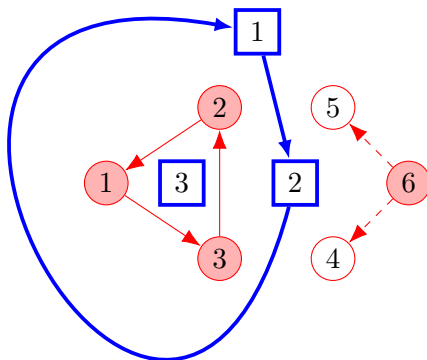


Figure 7: A counterexample to the connected subgraph formulation of Williams.

We note that even in Figure 1 of Williams' paper [14] the choice of primal and dual roots violates the Root Rule, and so the x and y variables may not select a primal arborescence. However, the vertices selected by the u variables will luckily induce a connected subgraph for that instance. This is why we give a different counterexample—to illustrate that the selected vertices may be disconnected.

It can be shown that this formulation for vertex-induced connected subgraphs

is correct when following the Root Rule. The proof of this is not difficult in light of Theorem 1 and is omitted.

Acknowledgements

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References

- [1] H. Ahmadi and J.R. Martí, *Mathematical representation of radiality constraint in distribution system reconfiguration problem*, Int. J. Electrical Power Energy Syst. **64** (2015), 293–299.
- [2] G. Braun, S. Fiorini, and S. Pokutta, *Average case polyhedral complexity of the maximum stable set problem*, Math. Program. **160** (2016), 407–431.
- [3] M. Conforti, V. Kaibel, M. Walter, and S. Weltge, *Subgraph polytopes and independence polytopes of count matroids*, Oper. Res. Lett. **43** (2015), 457–460.
- [4] R. Diestel, *Graph Theory* Vol. 173 of *Graduate Texts in Mathematics*, Springer, 2010.
- [5] Y. Faenza, S. Fiorini, R. Grappe, and H.R. Tiwary, *Extended formulations, nonnegative factorizations, and randomized communication protocols*, Math. Program. **153** (2015), 75–94.
- [6] S. Fiorini, T. Huynh, G. Joret, and K. Pashkovich, *Smaller extended formulations for the spanning tree polytope of bounded-genus graphs*, Discr. Comput. Geom. **57** (2017), 757–761.
- [7] S. Iwata, N. Kamiyama, N. Katoh, S. Kijima, and Y. Okamoto, *Extended formulations for sparsity matroids*, Math. Program. **158** (2016), 565–574.
- [8] V. Kaibel, J. Lee, M. Walter, and S. Weltge, *Extended formulations for independence polytopes of regular matroids*, Graphs Combin. **32** (2016), 1931–1944.
- [9] M. Kim and N. Xiao, *Contiguity-based optimization models for political redistricting problems*, Int. J. Appl. Geospatial Res. (IJAGR) **8** (2017), 1–18.
- [10] K. Pashkovich, *Extended formulations for combinatorial polytopes*, Ph.D. thesis, Otto-von-Guericke-Universität Magdeburg, 2012.

- [11] W.T. Tutte, *Graph Theory* Vol. 21 of *Encyclopedia of Mathematics and its Applications*, Addison-Wesley, 1984.
- [12] Y.c. Wang and H. Önal, *Designing connected nature reserve networks using a graph theory approach*, *Acta Ecologica Sinica* **31** (2011), 235–240.
- [13] J.C. Williams, *A linear-size zero-one programming model for the minimum spanning tree problem in planar graphs*, *Networks* **39** (2002), 53–60.
- [14] J.C. Williams, *A zero-one programming model for contiguous land acquisition*, *Geogr. Anal.* **34** (2002), 330–349.
- [15] N. Xiao, *An evolutionary algorithm for site search problems*, *Geogr. Anal.* **38** (2006), 227–247.