Political districting to minimize county splits

Maral Shahmizad and Austin Buchanan

School of Industrial Engineering & Management
Oklahoma State University

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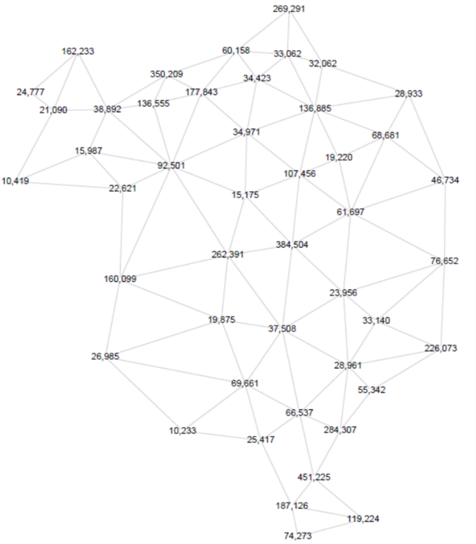
My Coauthor

Maral Shahmizad

Can you find 7 contiguous districts for South Carolina such that each has population between 657,463 and 664,070? Provide a solution or an argument for infeasibility.



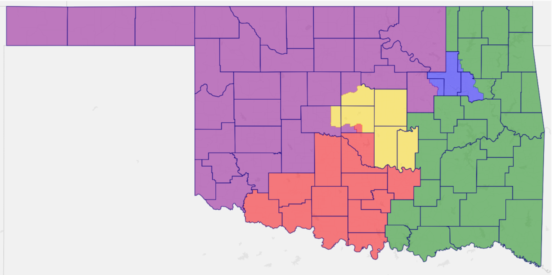
What is Political Districting?

Traditional criteria:

- Population balance
- Voting Rights Act
- Contiguity
- Compactness
- Political subdivisions
- . . .

Emerging criteria:

- Partisan fairness or proportionality
- Competitiveness
- . . .



Oklahoma congressional districts (2022)

Districting is Hard

Optimization is a Work in Progress

Photo: Melanie Acevedo

George Box said:

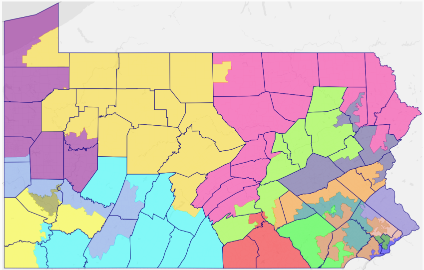
All models are wrong, but some are useful.

Optimizers should know:

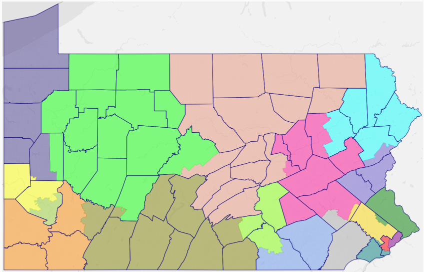
All optimization models are wrong, but some are useful.

Republican Gerrymandering in Pennsylvania

League of Women Voters of Pennsylvania v. the Commonwealth of Pennsylvania



Overturned districts (2013-2018)
28 split counties, 37 county splits



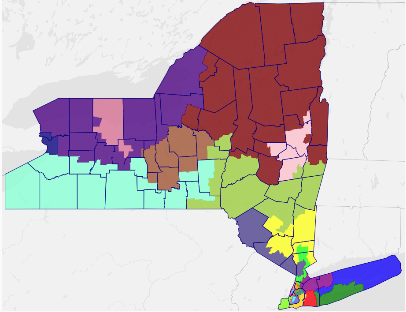
Court-mandated districts (2018)
13 split counties, 17 county splits

Quote from Pennsylvania Supreme Court:

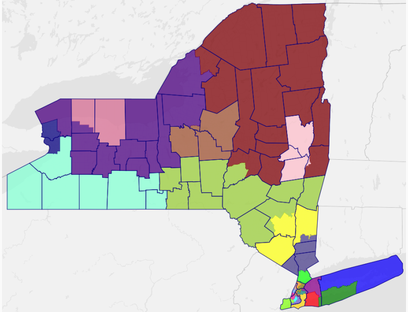
[The new plan shall] not divide any county, city, incorporated town, borough, township, or ward, except where necessary to ensure equality of population.

Democratic Gerrymandering in New York

Harkenrider v. Hochul



Overturned districts (2022)
34 split counties, 56 county splits



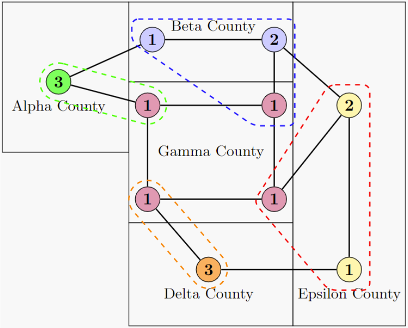
Court-mandated districts (2022)
16 split counties, 26 county splits

Quote from Special Master Jonathan Cervas:

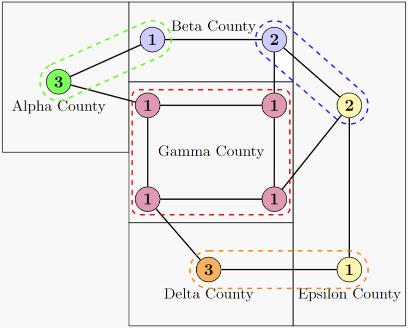
While I was quite successful in limiting the number of counties and cities that were split, some splits are simply inevitable. . . I can assure you that if yours was split it was not because of any kind of animus but was essentially due to the ***mathematical necessity of splitting some units****.* (Bold added)

Research Question

What is the **minimum number of county splits** possible (in a contiguous and population-balanced districting plan)?



1 split county, 3 county splits



2 split counties, 2 county splits

Answers from the Literature

John Nagle (2022) says:

The minimum number of county splits equals the number of districts minus one.

Dave’s Redistricting App (2023) says:

The minimum number of county splits is at most the number of districts minus one.

McCartan and Imai (2020) say:

Our algorithm generates plans with (number of districts minus one) county splits.

Autry, Carter, Herschlag, Hunter, and Mattingly (2021) say:

Our algorithm generates plans with (number of districts minus one) split counties.

Carter, Hunter, Teague, Herschlag, and Mattingly (2020) say:

The minimum number of county splits equals the number of districts minus the maximum number of county clusters.

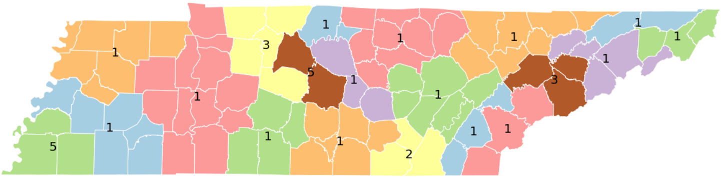
County Clustering

Suppose we want k districts, each with a population between L and U.

Definition (Carter et al., 2020)

Let G_C be the county-level graph. A county clustering is a partition $(C_1, C_2, \ldots, C_q)$ of the counties along with associated cluster sizes $(k_1, k_2, \ldots, k_q)$ such that:

1. the cluster sizes $(k_1, k_2, \ldots, k_q)$ are positive integers that sum to k;
2. each cluster C_j is contiguous, i.e., induces a connected subgraph of G_C;
3. each cluster C_j satisfies population balance, i.e., $Lk_j \leq p(C_j) \leq Uk_j$.

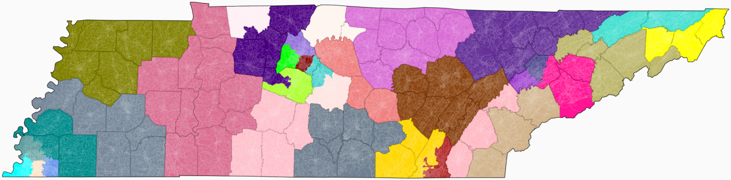


A **maximum** county clustering for the Tennessee State Senate
($k = 33$ districts, $q = 20$ county clusters)

County Clustering

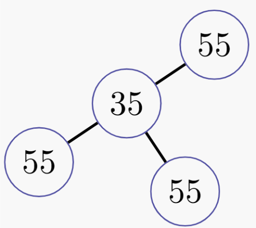
Theorem (Carter et al., 2020)

The minimum number of county splits equals the number of districts minus the maximum number of county clusters, "except in rare circumstances which affect the optimal districting".



A districting plan for the Tennessee State Senate
($k = 33$ districts, $q = 20$ county clusters, $k - q = 13$ county splits)

Enter the claw...



Divide this county-level graph into $k = 2$ districts
with populations between $L = 95$ and $U = 105$.

The maximum number of county clusters is $q = 1$.
Carter's theorem suggests $k - q = 2 - 1 = 1$ split.

But, we actually need 2 split counties and 2 county splits!

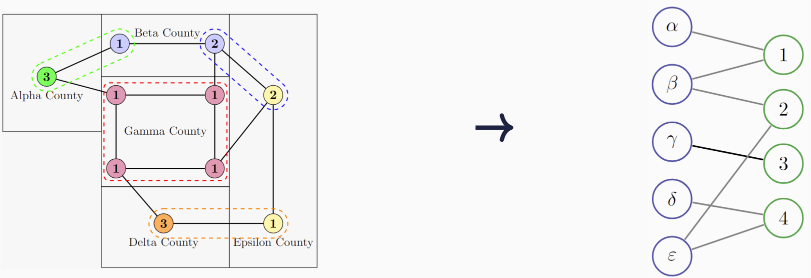
Definition (Split Duality)

A districting instance exhibits **weak split duality** if the minimum number of county splits is at least the number of districts minus the maximum number of county clusters. It exhibits **strong split duality** if equality also holds.

Proposition

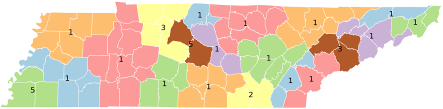
Weak split duality always holds. Strong split duality does not always hold.

Proof. Take a districting plan with a minimum number s of splits. Construct the county-district incidence graph. It has $n = |C| + k$ vertices and $m = |C| + s$ edges, so its number of connected components is at least $n - m = (|C| + k) - (|C| + s) = k - s$. Construct a cluster from each.

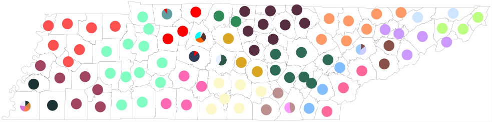


Our overall approach for min-split districting has three steps:

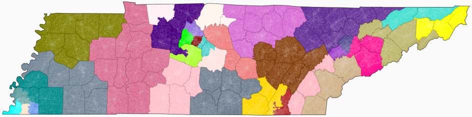
1. **Cluster**. Find a maximum number of county clusters $(C_1, C_2, \ldots, C_q)$ with associated cluster sizes $(k_1, k_2, \ldots, k_q)$.



2. **Sketch**. For each cluster C_j, sketch a plan using $k_j - 1$ county splits.



3. **Detail**. For each cluster C_j, find a plan that abides by the sketch's *support*.



If **Cluster** is successful, then $k - q$ is a lower bound. If **Sketch** and **Detail** are successful, then $k - q$ is an upper bound. If all, then **optimal!**

MIP for Cluster

- $x_{ij} = \begin{cases} 1 & \text{if county } i \text{ is assigned to the cluster rooted at county } j \\ 0 & \text{otherwise} \end{cases}$
- y_j = size of the cluster rooted at county j

$$\begin{aligned}
\max \quad & \sum_{j \in C} x_{jj} & \\
\text{s.t.} \quad & \sum_{j \in C} x_{ij} = 1 & \forall i \in C \\
& \sum_{j \in C} y_j = k & \\
& C_j = \{i \in C \mid x_{ij} = 1\} \text{ is connected} & \forall j \in C \\
& L y_j \leq \sum_{i \in C} p_i x_{ij} \leq U y_j & \forall j \in C \\
& x_{ij} \leq x_{jj} & \forall i, j \in C \\
& x_{ij} \in \{0, 1\} & \forall i, j \in C \\
& y_j \in \mathbb{Z}_+ & \forall j \in C.
\end{aligned}$$

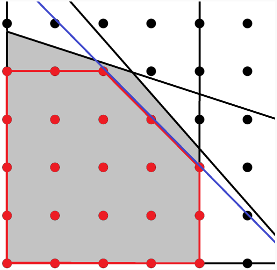
Key implementation details for max cluster:

- MIP is slow out-of-the-box; needs computational tricks!
- Symmetry handling (asymmetric representatives)
- MIP-based construction heuristic: carving à la McCartan and Imai (2020)
- MIP-based local search: t-opt recombination à la DeFord et al. (2021)
- Valid inequalities to strengthen the linear programming relaxation (!)

Theorem (Rounding Inequalities)

Let t be a positive integer, and let j be a county. The following inequality is valid.

$$\sum_{i \in C} \left\lfloor \frac{tp_i}{U+1} \right\rfloor x_{ij} \leq ty_j - x_{jj}.$$



MIP for Sketch

- $x_{ij} = \begin{cases} 1 & \text{if some of county } i \text{ is assigned to district number } j \in [k] \\ 0 & \text{otherwise} \end{cases}$
- z_{ij} = the proportion of county i that is assigned to district number $j \in [k]$
- s_i = the number of times that county i is split

Constraints for splitting and population balance:

$$\sum_{i \in C} s_i = k - 1$$

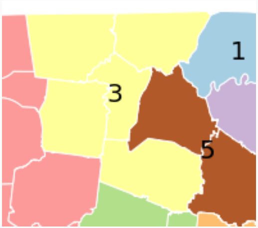
$$\sum_{j=1}^{k} x_{ij} = s_i + 1 \qquad \forall i \in C$$

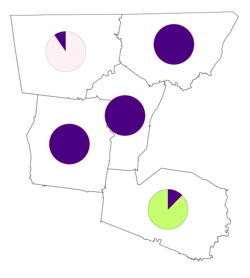
$$\sum_{j=1}^{k} z_{ij} = 1 \qquad \forall i \in C$$

$$L \leq \sum_{i \in C} p_i z_{ij} \leq U \qquad \forall j \in [k]$$

$$0 \leq z_{ij} \leq x_{ij} \qquad \forall i \in C,\ \forall j \in [k]$$

$$x_{ij} \in \{0, 1\} \qquad \forall i \in C,\ \forall j \in [k].$$





MIP for Sketch

- $y_{ej} = \begin{cases} 1 & \text{if county edge } e \text{ is preserved in district number } j \in [k] \\ 0 & \text{otherwise} \end{cases}$

Edge consistency constraints:

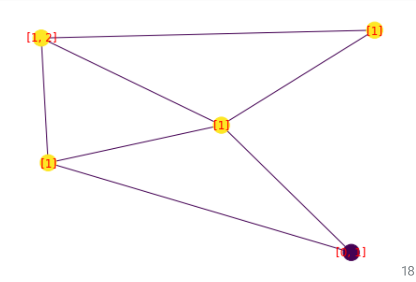
$$\begin{aligned} y_{ej} &\leq x_{ij} & \forall i \in e \in E(C),\ \forall j \in [k] \\ y_{ej} &\geq \sum_{i \in e} x_{ij} - 1 & \forall e \in E(C),\ \forall j \in [k] \\ \sum_{j=1}^{k} y_{ej} &\leq 1 & \forall e \in E(C) \\ y_{ej} &\in \{0, 1\} & \forall e \in E(C),\ \forall j \in [k]. \end{aligned}$$

Each district has (# edges) $\geq$ (# nodes) - 1:

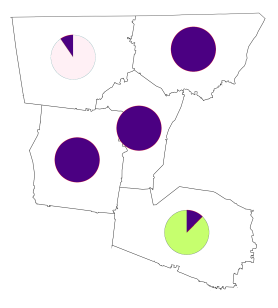
$$\sum_{e \in E(C)} y_{ej} \geq \sum_{i \in C} x_{ij} - 1 \qquad \forall j \in [k].$$

Objective: maximize preserved edges:

$$\max \sum_{e \in E(C)} \sum_{j=1}^{k} y_{ej}.$$



- Task: convert cluster's sketch into detailed plan



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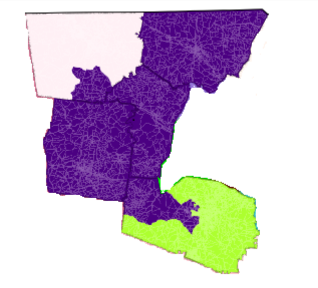


- Challenge: large tract- and block-level instances $G = (V, E)$
- Approach: capacitated k-means algorithm, similar to Hess et al. (1965), Bradley et al. (2000), Validi et al. (2022)

MIP for Detail

$$x_{ij} = \begin{cases} 1 & \text{if tract/block } i \text{ is assigned to district number } j \in [k] \\ 0 & \text{otherwise} \end{cases}$$

Solve this MIP, with better and better district means $(m_1, m_2, \ldots, m_k)$, with additional contiguity and sketch support constraints.

$$\min \sum_{i \in V} \sum_{j=1}^{k} (1 + p_i)\, \text{dist}(i, m_j)^2 x_{ij}$$

$$\sum_{j=1}^{k} x_{ij} = 1 \qquad \forall i \in V$$

$$L \leq \sum_{i \in V} p_i x_{ij} \leq U \qquad \forall j \in [k]$$

$$x_{ij} \in \{0, 1\} \qquad \forall i \in V,\ \forall j \in [k].$$

Setup:

- Maral's Desktop PC has Intel Xeon Processor E52630 v4 (10 cores, 2.2GHz, 3.1GHz Turbo) and 32 GB RAM. MIP solver is Gurobi v10.0.
- P.L. 94-171 data from Census; initial processing by Redistricting Data Hub; then by Daryl DeFord; # enacted splits calculated from 2022 plans
- $\pm 0.5\%$ deviation for congressional instances; $\pm 5\%$ deviation for legislative

Questions:

- Does strong split duality hold in practice?
- How does the minimum number of splits compare to enacted plans?
- How strong is the "obvious lower bound"?

Proposition

County c must be divided across $\lceil p_c/U \rceil$ or more districts, so at least this many splits:

$$\text{(obvious lower bound)} \quad \sum_{c \in C} \left(\lceil p_c/U \rceil - 1 \right).$$

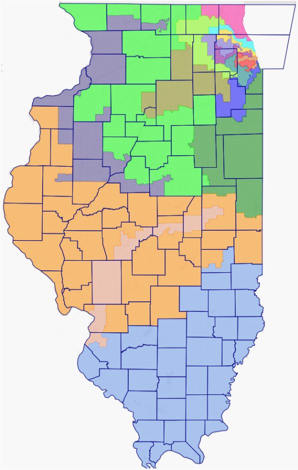
Congressional Results (1/2)

state	$\|C\|$	k	L	U	obvious LB	max clusters	split LB	min splits	enacted splits
AL	67	7	714,166	721,342	0	7	0	0	6
AR	75	4	749,117	756,645	0	4	0	0	3
AZ	15	9	790,639	798,584	6	2	7	7	15
CA	58	52	756,549	764,152	39	11	41	41	72
CO	64	8	718,106	725,322	1	6	2	2	20
CT	8	5	717,583	724,794	3	1	4	4	10
FL	67	28	765,375	773,067	10	9	19	19	31
GA	159	14	761,311	768,961	2	12	2	2	21
IA	99	4	793,605	801,580	0	4	0	0	0
ID	44	2	914,956	924,150	0	2	0	0	1
IL	102	17	749,909	757,445	7	8	9	9	53
IN	92	9	750,178	757,717	1	8	1	1	8
KS	105	4	730,798	738,142	0	4	0	0	4
KY	120	6	747,218	754,727	1	5	1	1	6
LA	64	6	772,412	780,174	0	6	0	0	15
MA	14	9	777,197	785,007	5	2	7	7	22
MD	24	8	768,293	776,013	3	4	4	4	9
ME	16	2	677,774	684,585	0	2	0	0	1
MI	83	13	771,304	779,055	4	9	4	4	21
MN	87	8	709,746	716,878	1	6	2	2	12
MO	115	8	765,518	773,210	1	7	1	1	10
MS	82	4	736,619	744,021	0	4	0	0	4

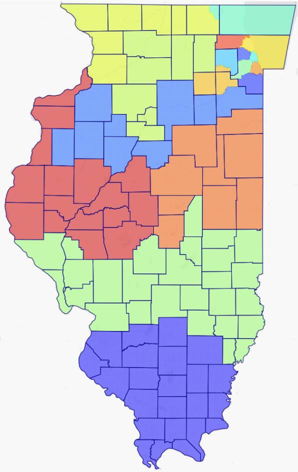
Congressional Results (2/2)

state	$\|C\|$	k	L	U	obvious LB	max clusters	split LB	min splits	enacted splits
MT	56	2	539,402	544,823	0	2	0	0	1
NC	100	14	741,943	749,398	2	11	3	3	13
NE	93	3	650,566	657,103	0	3	0	0	2
NH	10	2	685,321	692,208	0	1	1	1	5
NJ	21	12	770,213	777,953	3	3	9	9	20
NM	33	3	702,312	709,369	0	3	0	0	10
NV	17	4	772,273	780,034	2	2	2	2	5
NY	62	26	773,087	780,855	13	8	18	18	26
OH	88	15	782,697	790,563	3	11	4	4	14
OK	77	5	787,912	795,829	1	4	1	1	7
OR	36	6	702,679	709,740	1	5	1	1	16
PA	67	17	761,041	768,689	4	10	7	7	17
RI	5	2	545,947	551,432	1	1	1	1	1
SC	46	7	727,548	734,859	0	6	1	1	10
TN	95	9	764,032	771,710	1	7	2	2	11
TX	254	38	763,153	770,821	19	19	19	19	59
UT	29	4	813,815	821,993	1	3	1	1	7
VA	133	11	780,749	788,595	1	9	2	2	11
WA	39	10	766,676	774,380	4	6	4	4	11
WI	72	8	733,032	740,398	1	7	1	1	13
WV	55	2	892,374	901,342	0	2	0	0	0

Examples

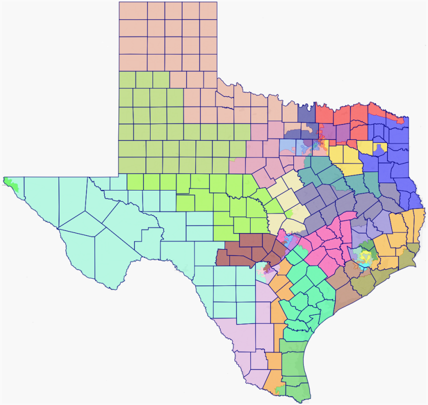


IL enacted (53 county splits)

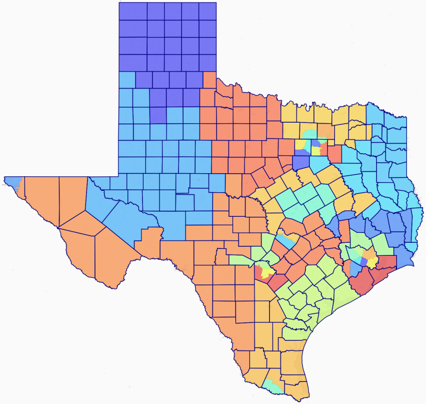


IL min-split (9 county splits)

Examples



TX enacted (59 county splits)



TX min-split (19 county splits)

State Senate Results (1/2)

state	|C|	k	L	U	obvious LB	max clusters	split LB	min splits	enacted splits
AK	30	20	34,837	38,503	12	7	13	13	19
AL	67	35	136,374	150,728	13	19	16	16	35
AR	75	35	81,742	90,345	14	21	14	14	51
AZ	15	30	226,465	250,302	22	6	24	24	44
CA	58	40	939,033	1,037,878	23	14	26	26	56
CO	64	35	156,716	173,211	22	13	22	22	42
CT	8	36	95,157	105,173	31	4	32	32	49
DE	3	21	44,784	49,497	18	3	18	18	20
FL	67	40	511,532	565,377	18	16	24	24	32
GA	159	56	181,720	200,848	23	31	25	25	60
IA	99	50	60,618	66,997	20	29	21	21	46
ID	44	35	49,919	55,173	19	14	21	21	25
IL	102	59	206,304	228,019	39	20	39	39	135
IN	92	50	128,926	142,496	20	28	22	22	48
KS	105	40	69,775	77,119	19	21	19	19	36
KY	120	38	112,646	124,503	11	26	12	12	21
LA	64	39	113,459	125,401	20	18	21	21	77
MA	14	40	166,961	184,535	31	6	34	34	59
MD	24	47	124,859	138,001	35	10	37	37	45
ME	16	35	36,979	40,870	24	8	27	27	40
MI	83	38	251,934	278,452	19	18	20	20	64
MN	87	67	80,913	89,430	38	26	41	41	100
MO	115	34	171,976	190,078	14	20	14	14	16
MS	82	52	54,101	59,795	19	29	23	23	64

State Senate Results (2/2)

state	$\|C\|$	k	L	U	obvious LB	max clusters	split LB	min splits	enacted splits
MT	56	50	20,601	22,768	30	19	31	31	56
NC	100	50	198,349	219,227	20	28	22	22	24
ND	53	47	15,748	17,405	28	18	29	29	49
NE	93	49	38,030	42,032	26	21	28	28	37
NH	10	24	54,528	60,266	19	4	20	20	40
NJ	21	40	220,614	243,836	28	10	30	30	56
NM	33	42	47,897	52,938	28	13	29	29	64
NV	17	21	140,447	155,230	17	3	18	18	21
NY	62	63	304,623	336,687	42	20	43	43	66
OH	88	33	339,682	375,436	12	20	13	13	20
OK	77	48	78,363	86,610	22	25	23	23	59
OR	36	30	134,180	148,303	17	11	19	19	47
PA	67	50	247,052	273,056	26	23	27	27	47
RI	5	38	27,435	30,322	33	3	35	35	41
SC	46	46	105,707	116,833	26	16	30	30	68
SD	66	35	24,067	26,600	17	16	19	19	29
TN	95	33	198,949	219,890	13	20	13	13	15
TX	254	31	893,169	987,186	12	19	12	12	41
UT	29	29	107,174	118,455	22	7	22	22	41
VA	133	40	204,996	226,574	16	24	16	16	34
VT	14	30	20,365	22,507	23	6	24	24	18
WA	39	49	149,389	165,113	34	13	36	36	59
WI	72	33	169,668	187,527	12	20	13	13	73
WV	55	17	100,238	110,788	2	13	4	4	13
WY	23	30	18,267	20,189	17	10	20	20	25

State House Results (1/2)

state	$\|C\|$	k	L	U	obvious LB	max clusters	split LB	min splits	enacted splits
AK	30	40	17,419	19,251	28	10	30	30	39
AL	67	105	45,458	50,242	71	29	76	76	115
AR	75	100	28,610	31,621	61	33	67	67	128
AZ	15	30	226,465	250,302	22	6	24	24	44
CA	58	80	469,517	518,939	57	20	60	60	95
CO	64	65	84,386	93,267	46	18	47	47	73
CT	8	151	22,687	25,074	139	8	143	143	162
DE	3	41	22,938	25,352	38	2	39	39	40
FL	67	120	170,511	188,459	89	26	94	94	112
GA	159	180	56,536	62,486	118	57	123	123	209
IA	99	100	30,309	33,498	54	38	62	62	92
ID	44	35	49,919	55,173	19	14	21	21	25
IL	102	118	103,152	114,009	85	31	87	87	220
IN	92	100	64,463	71,248	55	39	61	61	129
KS	105	125	22,328	24,678	87	34	91	91	127
KY	120	100	42,806	47,311	47	45	55	55	80
LA	64	105	42,142	46,577	72	29	76	76	116
MA	14	160	41,741	46,133	145	10	150	150	182
MD	24	47	124,859	138,001	35	10	37	37	67
ME	16	151	8,572	9,473	137	11	140	140	166
MI	83	110	87,032	96,192	73	32	78	78	154
MN	87	134	40,457	44,715	92	34	100	100	176
MO	115	163	35,873	39,648	110	47	116	116	137
MS	82	122	23,060	25,486	79	36	86	86	181

State House Results (2/2)

state	$\|C\|$	k	L	U	obvious LB	max clusters	split LB	min splits	enacted splits
MT	56	100	10,301	11,384	74	22	78	78	99
NC	100	120	82,646	91,344	71	40	80	80	80
ND	53	47	15,748	17,405	28	18	29	29	53
NH	10	400	3,272	3,616	375	10	390	390	154
NJ	21	40	220,614	243,836	28	10	30	30	56
NM	33	70	28,738	31,762	52	15	55	55	86
NV	17	42	70,224	77,615	35	4	38	38	43
NY	62	150	127,942	141,408	115	26	124	124	179
OH	88	99	113,228	125,145	57	35	64	64	77
OK	77	101	37,242	41,161	67	30	71	71	134
OR	36	60	67,090	74,151	44	13	47	47	79
PA	67	203	60,851	67,255	158	39	164	164	186
RI	5	75	13,901	15,363	70	4	71	71	75
SC	46	124	39,214	43,341	94	24	100	100	145
SD	66	35	24,067	26,600	17	16	19	19	31
TN	95	99	66,317	73,296	55	36	63	63	74
TX	254	150	184,589	204,018	99	50	100	100	101
UT	29	75	41,441	45,802	59	12	63	63	72
VA	133	100	81,999	90,629	55	38	62	62	98
VT	14	150	4,073	4,501	137	12	138	138	118
WA	39	49	149,389	165,113	34	13	36	36	59
WI	72	99	56,556	62,509	63	30	69	69	159
WV	55	100	17,041	18,834	70	24	76	76	89
WY	23	60	9,134	10,094	44	11	49	49	56

What About 1-Person Deviation?

John Nagle (2022) writes:

Forcing districts to satisfy a 1-person deviation makes it “highly probable that the minimum number of county splits is uniquely given as the number of districts minus one.”

Autry, Carter, Herschlag, Hunter, Mattingly (2021) write:

“It is reasonable to assume that there is no subset of counties that perfectly can accommodate a subset of the congressional districts. . . [which] may be used to demonstrate that $k - 1$ splits is optimal.”

state	$\lvert C \rvert$	CD	SS	SH	state	$\lvert C \rvert$	CD	SS	SH	state	$\lvert C \rvert$	CD	SS	SH
AK	30	■	✗	✗	MA	14	✗	✗	✗	OK	77	✓	✓	✓
AL	67	✓	✓	✓	MD	24	✗	✗	✗	OR	36	✓	✓	✓
AR	75	✓	✓	✓	ME	16	✗	✗	✓	PA	67	✓	✓	✓
AZ	15	✗	✗	✗	MI	83	✓	✓	✓	RI	5	✗	✗	✗
CA	58	✓	✓	✓	MN	87	✓	✓	✓	SC	46	✓	✓	✓
CO	64	✓	✓	✓	MO	115	✓	✓	✓	SD	66	■	✓	✓
CT	8	✗	✗	✓	MS	82	✓	✓	✓	TN	95	✓	✓	✓
DE	3	■	✗	✗	MT	56	✓	✓	✓	TX	254	✓	✓	✓
FL	67	✓	✓	✓	NC	100	✓	✓	✓	UT	29	✗	✓	✓
GA	159	✓	✓	✓	ND	53	■	✓	✓	VA	133	✓	✓	✓
IA	99	✓	✓	✓	NE	93	✓	✓	■	VT	14	■	✗	✓
ID	44	✓	✓	✓	NH	10	✗	✗	✓	WA	39	✓	✓	✓
IL	102	✓	✓	✓	NJ	21	✗	✗	✗	WI	72	✓	✓	✓
IN	92	✓	✓	✓	NM	33	✓	✓	✓	WV	55	✓	✓	✓
KS	105	✓	✓	✓	NV	17	✗	✗	✗	WY	23	■	✓	✓
KY	120	✓	✓	✓	NY	62	✓	✓	✓					
LA	64	✓	✓	✓	OH	88	✓	✓	✓					

Summary

- Many state constitutions say to preserve counties (cities, towns, ...).
- Prior to our work, the minimum number of splits was not known.
- We propose first exact approach: Cluster-Sketch-Detail.
- **Carter et al. (2020) are right!** Strong split duality *does* hold in practice.
- Many states' districting plans divide counties much more than necessary.

Disclaimer

- We do not claim that our computer maps should be enacted in practice.
- They do not consider the Voting Rights Act or any laws that vary by state.
- Create your own maps using the county clusterings as starting points!

Future work

- Determine the VRA-constrained minimum number of splits.
- Determine the tradeoffs between splits and population deviation.

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