# Political districting to minimize cut edges

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# What is political districting?

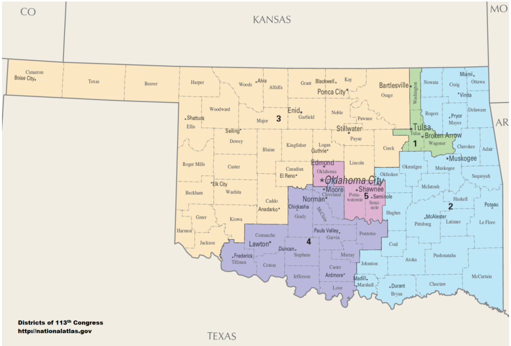


Figure: Oklahoma's congressional districts, 2013-now.

Traditional principles:

- Given # districts
- Population balance
    - $\pm 0.5\%$ congressional
    - $\pm 5.0\%$ legislative
- Contiguity
- Compactness
- Political subdivisions

# What is compact?

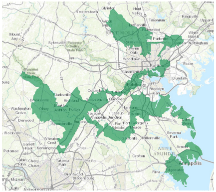


Figure: Maryland’s 3rd

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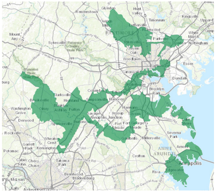


Figure: Maryland's 3rd

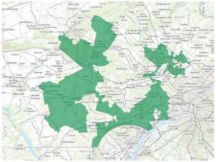

Figure: Pennsylvania's 7th

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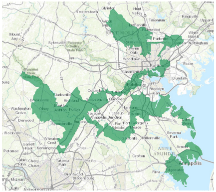

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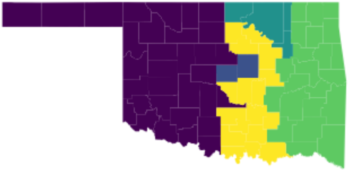
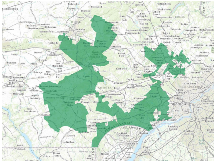


Figure: Oklahoma, minimizing moment-of-inertia



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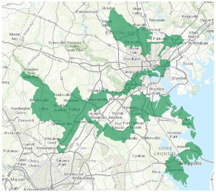

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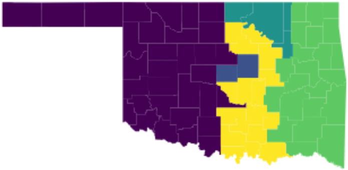

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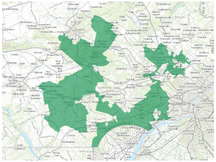

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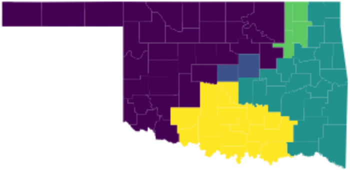

Figure: Oklahoma, minimizing cut edges

# What are cut edges?

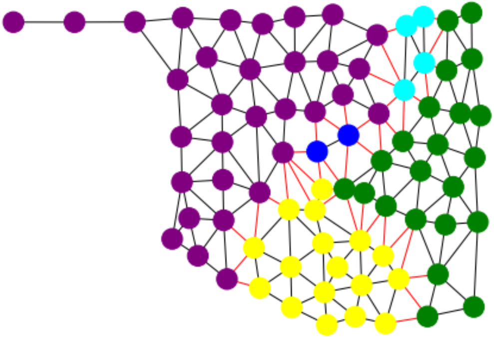

Figure: This plan cuts 40 edges in county-level graph $G = (V, E)$. Each of the $k = 5$ districts is contiguous and within population bounds $L = 746{,}519$ and $U = 754{,}021$.

# How to minimize cut edges in a MIP?

Variables:

$$x_{ij} = \begin{cases} 1 & \text{if vertex } i \in V \text{ is assigned to district } j \in [k] \\ 0 & \text{otherwise} \end{cases}$$

$$y_e = \begin{cases} 1 & \text{if edge } e \in E \text{ is cut} \\ 0 & \text{otherwise} \end{cases}$$

Labeling base model:

$$\min \sum_{e \in E} y_e \tag{1a}$$

$$x_{uj} - x_{vj} \leq y_e \qquad \forall e = \{u, v\} \in E,\ \forall j \in [k] \tag{1b}$$

$$\sum_{j \in [k]} x_{ij} = 1 \qquad \forall i \in V \tag{1c}$$

$$L \leq \sum_{i \in V} p_i x_{ij} \leq U \qquad \forall j \in [k] \tag{1d}$$

$$x \in \{0,1\}^{|V| \times k},\ y \in \{0,1\}^{|E|}. \tag{1e}$$

(Lacks contiguity constraints. . . )

## Results for *naïve* Labeling model

| state | $n$ | $k$ | B&B nodes | LP obj | MIP obj | time | contiguous? |
|---|---|---|---|---|---|---|---|
| ME | 16 | 2 | 1 | 0 | 8 | 0.07 | no |
| NM | 33 | 3 | 92 | 0 | 17 | 0.19 | yes |
| ID | 44 | 2 | 52 | 0 | 10 | 0.14 | yes |
| WV | 55 | 3 | 6,524 | 0 | 20 | 2.24 | no |
| LA | 64 | 6 | 2,540,158 | 0 | [36,45] | Time Limit | * |
| AL | 67 | 7 | 1,586,159 | 0 | [38,54] | Time Limit | * |
| AR | 75 | 4 | 266,038 | 0 | 32 | 203.85 | no |
| OK | 77 | 5 | 76,338 | 0 | 39 | 132.88 | no |
| MS | 82 | 4 | 836,663 | 0 | 32 | 457.61 | no |
| NE | 93 | 3 | 2,104 | 0 | 19 | 2.86 | yes |
| IA | 99 | 4 | 529,067 | 0 | 33 | 716.04 | yes |
| KS | 105 | 4 | 320,451 | 0 | 31 | 425.73 | no |

* If given more time, these approaches would return "no".

# An alternative MIP

Variables of Hess et al. (1965):

$$x_{ij} = \begin{cases} 1 & \text{if vertex } i \in V \text{ is assigned to the district rooted at vertex } j \in V \\ 0 & \text{otherwise} \end{cases}$$

Hess base model:

$$\min \sum_{e \in E} y_e \tag{2a}$$

$$x_{uj} - x_{vj} \leq y_e \qquad \forall e = \{u, v\} \in E, \ \forall j \in V \tag{2b}$$

$$\sum_{j \in V} x_{ij} = 1 \qquad \forall i \in V \tag{2c}$$

$$L x_{jj} \leq \sum_{i \in V} p_i x_{ij} \leq U x_{jj} \qquad \forall j \in V \tag{2d}$$

$$\sum_{j \in V} x_{jj} = k \tag{2e}$$

$$x_{ij} \leq x_{jj} \qquad \forall i, j \in V \tag{2f}$$

$$x \in \{0,1\}^{|V| \times |V|}, y \in \{0,1\}^{|E|}. \tag{2g}$$

(Lacks contiguity constraints...)

# Results for *naïve* Hess model

| state | $n$ | $k$ | B&B nodes | LP obj | MIP obj | time |
|---|---|---|---|---|---|---|
| ME | 16 | 2 | 1,480 | 0.27 | 8 | 2.32 |
| NM | 33 | 3 | 49,106 | 0.48 | 17 | 526.52 |
| ID | 44 | 2 | 12,326 | 0.09 | 10 | 652.21 |
| WV | 55 | 3 | 9,301 | 0.14 | [1,20] | Time Limit |
| LA | 64 | 6 | 3,708 | 0.77 | [5,48] | Time Limit |
| AL | 67 | 7 | 2,018 | 0.94 | [14,55] | Time Limit |
| AR | 75 | 4 | 25 | 0.27 | [3,45] | Time Limit |
| OK | 77 | 5 | 25 | 0.36 | [4,49] | Time Limit |
| MS | 82 | 4 | 1 | 0.25 | [2,62] | Time Limit |
| NE | 93 | 3 | 1 | 0.11 | [1,30] | Time Limit |
| IA | 99 | 4 | 7 | 0.18 | [1,51] | Time Limit |
| KS | 105 | 4 | 1 | 0.18 | [1,53] | Time Limit |

# Tricks from the MIP arsenal

1. Heuristic warm start, via GerryChain
2. Symmetry handling
   - Labeling: partitioning orbitope, see Faenza and Kaibel (2009)
   - Hess: asymmetric representatives (“diagonal fixing”)
3. Variable fixing
   - $L$-fixing
   - $U$-fixing
4. Contiguity constraints
   - CUT: $a, b$-separator constraints, see Oehrlein and Haunert (2017)
   - LCUT: length-$U$ $a, b$-separator constraints, see Validi et al. (2021)
   - SHIR: flow-based model of Shirabe (2005, 2009)
   - SCF: flow-based model of Hojny et al. (2021)
5. Extended formulation for objective, see Ferreira et al. (1996)

# Symmetry handling for Hess

Asymmetric representatives: pick $(v_1, v_2, \ldots, v_n)$ and fix $x_{ij} = 0$ if $i$ comes before $j$.

| County | Population | # | 19 | 20 | 27 | 23 | 15 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 16 | 17 | 18 | 21 | 22 | 24 | 25 | 26 | 28 | 29 | 30 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Bernalillo | 662,564 | 19 | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Doña Ana | 209,233 | 20 | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Santa Fe | 144,170 | 27 | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Rio Arriba | 40,246 | 23 | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Cibola | 27,213 | 15 | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Harding | 695 | 1 | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Sierra | 11,988 | 2 | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Lea | 64,727 | 3 | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Guadalupe | 4,687 | 4 | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Torrance | 16,383 | 5 | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Grant | 29,514 | 6 | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Otero | 63,797 | 7 | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| San Juan | 130,044 | 8 | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Roosevelt | 19,846 | 9 | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Curry | 48,376 | 10 | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Taos | 32,937 | 11 | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Hidalgo | 4,894 | 12 | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Eddy | 53,829 | 13 | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| De Baca | 2,022 | 14 | | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Quay | 9,041 | 16 | | | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Colfax | 13,750 | 17 | | | | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D |
| Los Alamos | 17,950 | 18 | | | | | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D |
| Chaves | 65,645 | 21 | | | | | | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D |
| Valencia | 76,569 | 22 | | | | | | | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D |
| San Miguel | 29,393 | 24 | | | | | | | | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D | D |
| Catron | 3,725 | 25 | | | | | | | | | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D | D |
| Sandoval | 131,561 | 26 | | | | | | | | | | | | | | | | | | | | | | | | | | | | D | D | D | D | D | D |
| Socorro | 17,866 | 28 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | D | D | D | D | D |
| Lincoln | 20,497 | 29 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | D | D | D | D |
| McKinley | 71,492 | 30 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | D | D | D |
| Luna | 25,095 | 31 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | D | D |
| Mora | 4,881 | 32 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | D |
| Union | 4,549 | 33 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

For NM, "diagonal fixing" removes $528 = (n^2 - n)/2$ of the $1{,}089 = n^2$ Hess variables.

# $L$-fixing (light) for Hess

After diagonal fixing, only some vertices can be assigned to vertex $j \in V$:

$$V_j = \{i \in V \mid \text{pos}(i) \geq \text{pos}(j)\}.$$

Vertex $j$ cannot root a population-feasible district if $\sum_{i \in V_j} p_i < L$, so $x_{jj} = 0$.

| County | Population | # | 19 | 20 | 27 | 23 | 15 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 16 | 17 | 18 | 21 | 22 | 24 | 25 | 26 | 28 | 29 | 30 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Bernalillo | 662,564 | 19 | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Doña Ana | 209,233 | 20 | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Santa Fe | 144,170 | 27 | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Rio Arriba | 40,246 | 23 | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Cibola | 27,213 | 15 | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Harding | 695 | 1 | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Sierra | 11,988 | 2 | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Lea | 64,727 | 3 | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Guadalupe | 4,687 | 4 | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Torrance | 16,383 | 5 | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Grant | 29,514 | 6 | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Otero | 63,797 | 7 | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| San Juan | 130,044 | 8 | | | | | | | | | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Roosevelt | 19,846 | 9 | | | | | | | | | | | | | | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Curry | 48,376 | 10 | | | | | | | | | | | | | | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
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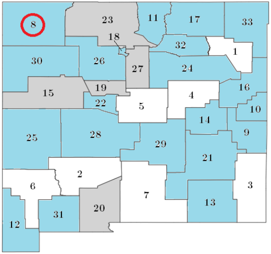
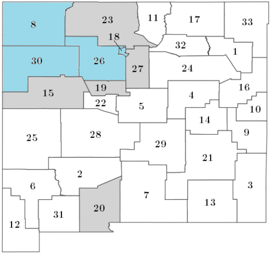
| Hidalgo | 4,894 | 12 | | | | | | | | | | | | | | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Eddy | 53,829 | 13 | | | | | | | | | | | | | | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| De Baca | 2,022 | 14 | | | | | | | | | | | | | | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Quay | 9,041 | 16 | | | | | | | | | | | | | | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Colfax | 13,750 | 17 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D |
| Los Alamos | 17,950 | 18 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D |
| Chaves | 65,645 | 21 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D |
| Valencia | 76,569 | 22 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D |
| San Miguel | 29,393 | 24 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D |
| Catron | 3,725 | 25 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D |
| Sandoval | 131,561 | 26 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D |
| Socorro | 17,866 | 28 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D |
| Lincoln | 20,497 | 29 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D |
| McKinley | 71,492 | 30 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D |
| Luna | 25,095 | 31 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D |
| Mora | 4,881 | 32 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D |
| Union | 4,549 | 33 | | | | | | | | | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L |

After diagonal fixing, only vertices $V_8$ can still be assigned to vertex $j = 8$:




For contiguity-constrained problems, we can refine $V_8$ to reachable vertices $S_8$:



# $L$-fixing for Hess

When enforcing contiguity, we can refine $V_j$ to reachable vertices in $G[V_j]$:

$$S_j = \{i \in V_j \mid \text{there is an } i,j\text{-path in } G[V_j]\}$$

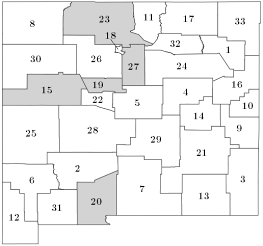
Vertex $j$ cannot root a <u>contiguous</u>, pop.-feasible district if $\sum_{i \in S_j} p_i < L$, so $x_{jj} = 0$.

| County | Population | # | 19 | 20 | 27 | 23 | 15 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 16 | 17 | 18 | 21 | 22 | 24 | 25 | 26 | 28 | 29 | 30 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Bernalillo | 662,564 | 19 |  | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Doña Ana | 209,233 | 20 |  |  | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Santa Fe | 144,170 | 27 |  |  |  | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Rio Arriba | 40,246 | 23 |  |  |  |  | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Cibola | 27,213 | 15 |  |  |  |  |  | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Harding | 695 | 1 |  |  |  |  |  | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Sierra | 11,988 | 2 |  |  |  |  |  | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Lea | 64,727 | 3 |  |  |  |  |  | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Guadalupe | 4,687 | 4 |  |  |  |  |  | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Torrance | 16,383 | 5 |  |  |  |  |  | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Grant | 29,514 | 6 |  |  |  |  |  | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Otero | 63,797 | 7 |  |  |  |  |  | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| San Juan | 130,044 | 8 |  |  |  |  |  | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Roosevelt | 19,846 | 9 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Curry | 48,376 | 10 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Taos | 32,937 | 11 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Hidalgo | 4,894 | 12 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Eddy | 53,829 | 13 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| De Baca | 2,022 | 14 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Quay | 9,041 | 16 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Colfax | 13,750 | 17 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D |
| Los Alamos | 17,950 | 18 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D |
| Chaves | 65,645 | 21 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D |
| Valencia | 76,569 | 22 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D |
| San Miguel | 29,393 | 24 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D |
| Catron | 3,725 | 25 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D |
| Sandoval | 131,561 | 26 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D |
| Socorro | 17,866 | 28 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D |
| Lincoln | 20,497 | 29 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D |
| McKinley | 71,492 | 30 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D |
| Luna | 25,095 | 31 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D |
| Mora | 4,881 | 32 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D |
| Union | 4,549 | 33 |  |  |  |  |  | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L |

# Getting the most out of $L$-fixing

## Lemma

*Suppose $B \subseteq V$ is such that every component of $G[B]$ has population less than $L$. If $B$ is placed at the back of the ordering, then every vertex $j \in B$ will be $L$-fixed.*




## Theorem

*If a solution to the following problem is placed at the back of the ordering, this maximizes the number of $L$-fixings.*

$$\max_{B \subseteq V} \{|B| : \textit{every component of } G[B] \textit{ has population less than } L\}.$$

## Solving the max $B$ problem

Variables:

$$x_{ij} = \begin{cases} 1 & \text{if vertex } i \in V \text{ is assigned to bin } j \in [q] \\ 0 & \text{otherwise} \end{cases}$$

$$b_i = \begin{cases} 1 & \text{if vertex } i \in V \text{ is chosen in } B \\ 0 & \text{otherwise} \end{cases}$$

Model:

$$\max \sum_{i \in V} b_i$$

$$\sum_{j \in [q]} x_{ij} = b_i \qquad \forall i \in V$$

$$\sum_{i \in V} p_i x_{ij} \leq L - 1 \qquad \forall j \in [q]$$

$$x_{uj} + b_v \leq 1 + x_{vj} \qquad \forall \{u,v\} \in E,\ \forall j \in [q]$$

$$x \in \{0,1\}^{|V| \times q},\ b \in \{0,1\}^{|V|}.$$

**Proposition**

*For our instances, $q = 2k$ bins suffice. Generally, this holds if $k \leq 99$ and ideal district population $\bar{p} = p(V)/k$ satisfies $L \geq 0.995\bar{p}$ and $\bar{p} \geq 39,800$.*

# $U$-fixing for Hess

If the population-weighted distance from $i$ to $j$ in $G[S_j]$ exceeds $U$, then $i$ and $j$ cannot belong to the same contiguous, population-feasible district, so $x_{ij} = 0$.

| County | Population | # | 19 | 20 | 27 | 23 | 15 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 16 | 17 | 18 | 21 | 22 | 24 | 25 | 26 | 28 | 29 | 30 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Bernalillo | 662,564 | 19 | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Doña Ana | 209,233 | 20 | U | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Santa Fe | 144,170 | 27 | U | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Rio Arriba | 40,246 | 23 | U | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Cibola | 27,213 | 15 | | | | | | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Harding | 695 | 1 | U | | | | | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Sierra | 11,988 | 2 | U | | | | | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Lea | 64,727 | 3 | U | | | | | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Guadalupe | 4,687 | 4 | | | | | | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Torrance | 16,383 | 5 | | | | | | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Grant | 29,514 | 6 | U | | | | | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Otero | 63,797 | 7 | U | | | | | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| San Juan | 130,044 | 8 | U | | | | | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Roosevelt | 19,846 | 9 | U | | | | | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Curry | 48,376 | 10 | U | | | | | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Taos | 32,937 | 11 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Hidalgo | 4,894 | 12 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Eddy | 53,829 | 13 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| De Baca | 2,022 | 14 | | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Quay | 9,041 | 16 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D | D |
| Colfax | 13,750 | 17 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D | D |
| Los Alamos | 17,950 | 18 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D | D |
| Chaves | 65,645 | 21 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D | D |
| Valencia | 76,569 | 22 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D | D |
| San Miguel | 29,393 | 24 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D | D |
| Catron | 3,725 | 25 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D | D |
| Sandoval | 131,561 | 26 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D | D |
| Socorro | 17,866 | 28 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D | D |
| Lincoln | 20,497 | 29 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D | D |
| McKinley | 71,492 | 30 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D | D |
| Luna | 25,095 | 31 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D | D |
| Mora | 4,881 | 32 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | D |
| Union | 4,549 | 33 | U | | | | | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L | L |

Diagonal-fixing, $L$-fixing, and $U$-fixing can be extended to the Labeling model.

## Variable fixing for Hess

| state | $n$ | $k$ | max $B$ model $\|B\|$ | time | How many variables are fixed? DFix | LFix | UFix | %X |
|---|---|---|---|---|---|---|---|---|
| ME | 16 | 2 | 13 | 0.03 | 120 | 91 | 0 | 82 |
| NM | 33 | 3 | 28 | 0.17 | 528 | 406 | 28 | 88 |
| ID | 44 | 2 | 41 | 0.05 | 946 | 861 | 0 | 93 |
| WV | 55 | 3 | 48 | 3.62 | 1,485 | 1,176 | 20 | 89 |
| LA | 64 | 6 | 53 | 30.68 | 2,016 | 1,431 | 58 | 86 |
| AL | 67 | 7 | 52 | >60.00 | 2,211 | 1,378 | 110 | 82 |
| AR | 75 | 4 | 64 | 24.01 | 2,775 | 2,080 | 28 | 87 |
| OK | 77 | 5 | 64 | 20.83 | 2,926 | 2,080 | 101 | 86 |
| MS | 82 | 4 | 69 | >60.00 | 3,321 | 2,415 | 20 | 86 |
| NE | 93 | 3 | 86 | 2.56 | 4,278 | 3,741 | 5 | 93 |
| IA | 99 | 4 | 85 | >60.00 | 4,851 | 3,655 | 52 | 87 |
| KS | 105 | 4 | 95 | 12.55 | 5,460 | 4,560 | 11 | 91 |
| NH | 295 | 2 | 283 | 7.61 | 43,365 | 40,186 | 0 | 96 |
| ID | 298 | 2 | 291 | 2.48 | 44,253 | 42,486 | 138 | 98 |
| ME | 358 | 2 | 349 | 3.19 | 63,903 | 61,075 | 0 | 98 |
| WV | 484 | 3 | 467 | >60.00 | 116,886 | 109,278 | 1,195 | 97 |
| NM | 499 | 3 | 485 | >60.00 | 124,251 | 117,855 | 184 | 97 |
| NE | 532 | 3 | 513 | >60.00 | 141,246 | 131,841 | 375 | 97 |
| UT | 588 | 4 | 556 | >60.00 | 172,578 | 154,846 | 918 | 95 |
| MS | 664 | 4 | 634 | >60.00 | 220,116 | 201,295 | 1,456 | 96 |
| AR | 686 | 4 | 653 | >60.00 | 234,955 | 213,531 | 1,982 | 96 |
| NV | 687 | 4 | 655 | >60.00 | 235,641 | 214,840 | 707 | 96 |

# Extended formulation for objective

| state | $n$ | $k$ | original LP | extended LP |
|---|---|---|---|---|
| ME | 16 | 2 | 2.74 | 3.62 |
| NM | 33 | 3 | 6.43 | 10.97 |
| ID | 44 | 2 | 2.53 | 4.27 |
| WV | 55 | 3 | 4.22 | 9.41 |
| LA | 64 | 6 | 5.73 | 22.30 |
| AL | 67 | 7 | 13.38 | 29.05 |
| AR | 75 | 4 | 6.40 | 15.92 |
| OK | 77 | 5 | 12.62 | 21.36 |
| MS | 82 | 4 | 4.16 | 12.88 |
| NE | 93 | 3 | 3.19 | 9.71 |
| IA | 99 | 4 | 3.58 | 14.66 |
| KS | 105 | 4 | 4.76 | 13.45 |

Variables:

$$z_e^j = \begin{cases} 1 & \text{if edge } e = \{u, v\},\ u < v, \text{ is cut because } u \to j \text{ but } v \not\to j \\ 0 & \text{otherwise} \end{cases}$$

Constraints (essentially) due to Ferreira et al. (1996):

$$\begin{aligned} & x_{uj} - x_{vj} \leq z_e^j && \forall e = \{u, v\} \in E,\ u < v, \forall j \in V \\ & y_e = \sum_{j \in V} z_e^j && \forall e \in E \\ & z_e^j \geq 0 && \forall e \in E,\ \forall j \in V. \end{aligned}$$

## Results when using MIP arsenal

| | | | Hess base model | | | Labeling base model | | |
|---|---|---|---|---|---|---|---|---|
| state | $n$ | $k$ | LCUT | SCF | SHIR | LCUT | SCF | SHIR |
| ME | 16 | 2 | 0.19 | 0.11 | 0.11 | 0.17 | 0.12 | 0.09 |
| NM | 33 | 3 | 0.28 | 0.12 | 0.12 | 0.09 | 0.06 | 0.06 |
| ID | 44 | 2 | 0.31 | 0.16 | 0.25 | 0.12 | 0.05 | 0.08 |
| WV | 55 | 3 | 3.04 | 1.69 | 2.14 | 1.11 | 1.03 | 1.23 |
| LA | 64 | 6 | 209.87 | 433.77 | 91.50 | 394.16 | 512.65 | 320.69 |
| AL | 67 | 7 | 1,125.44 | 1,200.22 | 1,252.13 | 1,019.40 | 1,269.54 | 956.12 |
| AR | 75 | 4 | 45.86 | 54.05 | 65.19 | 23.54 | 31.79 | 40.38 |
| OK | 77 | 5 | 9.43 | 10.06 | 12.89 | 8.50 | 4.11 | 6.78 |
| MS | 82 | 4 | 107.80 | 128.20 | 160.19 | 54.80 | 32.06 | 70.84 |
| NE | 93 | 3 | 2.47 | 4.86 | 4.87 | 1.44 | 0.62 | 0.91 |
| IA | 99 | 4 | 72.61 | 96.34 | 192.49 | 29.79 | 35.59 | 79.24 |
| KS | 105 | 4 | 62.00 | 20.61 | 29.30 | 27.21 | 13.03 | 27.09 |
| NH | 295 | 2 | 110.69 | 102.73 | 143.98 | 3.53 | 2.83 | 12.58 |
| ID | 298 | 2 | 22.43 | 33.25 | 23.84 | 1.77 | 1.02 | 1.70 |
| ME | 358 | 2 | 80.54 | 70.52 | 49.70 | 2.72 | 1.29 | 1.66 |
| WV | 484 | 3 | 1,142.36 | 656.71 | 2,239.97 | 158.64 | 151.84 | 334.49 |
| NM | 499 | 3 | 695.74 | 385.64 | TL | 24.38 | 96.35 | 62.80 |
| NE | 532 | 3 | TL | TL | TL | 163.78 | 207.06 | 392.56 |
| UT | 588 | 4 | TL | TL | TL | TL | TL | TL |
| MS | 664 | 4 | TL | TL | TL | TL | TL | TL |
| AR | 686 | 4 | TL | TL | TL | TL | TL | TL |
| NV | 687 | 4 | TL | TL | TL | TL | TL | TL |

LCUT (Validi et al., 2021), SCF (Hojny et al., 2021), SHIR (Shirabe, 2005; 2009)

# Conclusion

- Cut edges is a nice way to measure district compactness
- Cut edge minimization is slow using out-of-the-box MIPs
- Optimal solutions for $n \leq 532$ using tricks from the MIP arsenal
  - Heuristic warm start, via GerryChain
  - Symmetry handling (partitioning orbitope, diagonal fixing)
  - Variable fixing ($L$-fixing, $U$-fixing)
  - Contiguity constraints (LCUT, SHIR, SCF)
  - Extended formulation for objective
- Work generalizes to minimize district perimeters (extend to Polsby-Popper?)
- We make no claims that our maps are "good" or legal

Paper and Code

OR Redistricting Resources

# References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