# Extended Formulations for Vertex Cover

Austin Buchanan

*School of Industrial Engineering & Management, Oklahoma State University*

**Abstract**

The vertex cover polytopes of graphs do not admit polynomial-size extended formulations. This motivates the search for polyhedral analogues to approximation algorithms and fixed-parameter tractable (FPT) algorithms. In this paper, we take the FPT approach and study the $k$-vertex cover polytope (the convex hull of vertex covers of size $k$). Our main result is that there are extended formulations of size $O(1.47^k + kn)$. We also provide FPT extended formulations for solutions of size $k$ to instances of $d$-HITTING SET.

*Keywords:* fixed-parameter tractable, extended formulation, vertex cover, independent set, cardinality constrained, hitting set

## 1. Introduction

Two of the most well-studied problems in combinatorial optimization are the minimum vertex cover problem and the maximum independent set problem. Their polyhedral representations, the vertex cover polytope $\mathrm{VC}(G)$ and the independent (or stable) set polytope $\mathrm{STAB}(G)$, have been the subject of numerous studies [27, 25, 26, 31], and can be defined as follows. For a graph $G = (V, E)$,

$$\mathrm{VC}(G) := \mathrm{conv.hull}\left\{x^S \mid S \subseteq V \text{ is a vertex cover for } G\right\}$$
$$\mathrm{STAB}(G) := \mathrm{conv.hull}\left\{x^S \mid S \subseteq V \text{ is an independent set in } G\right\}$$

where $x^S$ denotes the characteristic vector of $S$. These two polytopes are closely related to each other. Indeed, letting $\mathbf{1}$ be a vector of ones, it is easy to see that

$$\mathrm{VC}(G) = \{x \mid (\mathbf{1} - x) \in \mathrm{STAB}(G)\}.$$

One reason for studying these polytopes is that exact, partial, or even approximate descriptions of them by linear inequalities can be used to help solve integer programs with set packing constraints [27].

Due to the difficulty of the minimum vertex cover problem and the maximum independent set problem, $\mathrm{VC}(G)$ and $\mathrm{STAB}(G)$ are likely to be complicated. Indeed, they can have exponentially many facets even when $G$ is series-parallel, but, in this case, $\mathrm{VC}(G)$ and $\mathrm{STAB}(G)$ admit linear-size *extended formulations* [3]. (The size of an extended formulation is the number of linear inequalities in its description.)

However, small extended formulations for $\mathrm{VC}(G)$ and $\mathrm{STAB}(G)$ are the exception rather than the rule. For some classes of graphs, $2^{\Omega(\sqrt{n})}$ linear inequalities are needed [16], and a lower bound of $2^{\Omega(n)}$ has been conjectured [7]. This motivates the search for approximate or fixed-parameter tractable (FPT) extended formulations. The polyhedral approximability of $\mathrm{VC}(G)$ and $\mathrm{STAB}(G)$ was recently studied by Bazzi et al. [4]. In this paper, we take the FPT approach. An extended formulation is said to be FPT if its size is bounded above by $f(k)n^{O(1)}$, where $k$ is the parameter of choice, $f$ is a function that depends only on $k$, and $n$ is the input size.

To formalize our approach, consider the $k$-vertex cover polytope $\mathrm{VC}_k(G)$ of a graph $G$, which is the convex hull of vertex covers of $G$ that have size $k$.

$$\mathrm{VC}_k(G) := \mathrm{conv.hull}\left\{x^S \mid S \subseteq V \text{ is a vertex cover for } G \text{ and } |S| = k\right\}.$$

The problem of determining whether a graph has a vertex cover of size $k$, i.e., the $k$-vertex cover problem, is the prototypical problem in the FPT literature, so there are numerous structural results that can be used when developing extended formulations. For example, we use Damaschke's [15] refinement of Sam Buss's kernel for $k$-vertex cover [11] and decomposition ideas from Chen et al. [12].

### 1.1. Our contributions

Our main result is that $k$-vertex cover polytopes of $n$-vertex graphs admit extended formulations of size $O(1.47^k + kn)$. En route to proving this, we exhibit size $O(n)$ formulations for the case of graphs of maximum degree at most two. In its proof, we use known results regarding the cardinality-constrained matching polytope and $t$-perfect graphs.

Before showing this, we provide simple extended formulations for the hypergraph case. They are of size $O(d^k n)$ for $d$-uniform hypergraphs. (In a $d$-uniform hypergraph, each hyperedge has $d$ vertices.) This is equivalent to solutions of size $k$ to instances of $d$-HITTING SET. Our results hold when each hyperedge has *at most* $d$ vertices, but we state everything for $d$-uniform hypergraphs for sake of exposition. This implies size $O(2^k n)$ extended formulations for the standard vertex cover problem. This bound is worse than what we eventually will show, but it is easier to obtain.

*1.2. Preliminaries and related work*

**Definition 1.** *Let $P = \{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$ be a polyhedron. A polyhedron $Q \subseteq \mathbb{R}^d$ is said to be an* extension *for $P$ if $\mathrm{proj}_x(Q) = P$, where $\mathrm{proj}_x(Q) := \{x \mid \exists y : (x, y) \in Q(G)\}$. The* size *of an extension is the number of its facets.*

A particular representation of an extension by linear inequalities is called an extended formulation. Consult the surveys of Conforti et al. [14] and Kaibel [19] for some notable extended formulations.

**Definition 2.** *The* extension complexity *of a polyhedron $P$ is*

$$\mathrm{xc}(P) := \min\{\mathrm{size}(Q) \mid Q \text{ is an extension for } P\}.$$

We will use Balas' extended formulation for the union of polyhedra.

**Theorem 1** (Balas [1, 2]). *Consider $q$ polytopes $P^i \subseteq \mathbb{R}^n$, $i = 1, \ldots, q$ and write $P := \mathrm{conv.hull}\left(\bigcup_{i=1}^q P^i\right)$. Then, $\mathrm{xc}(P) \leq q + \sum_{i=1}^q \mathrm{xc}(P^i)$.*

Since the work of Yannakakis [34], there have been numerous advances showing that certain polytopes have high extension complexity. For example, polytopes associated with NP-hard problems such as the traveling salesman problem and the 0-1 knapsack problem admit no polynomial-size extended formulation [16, 28]—irrespective of whether P=NP. There are even polynomial-time solvable problems such as matching that admit no polynomial-size extended formulations [29]. While the matching polytope does not admit a polyhedral equivalent to an FPTAS [8], it does admit FPT extended formulations parameterized by the number of edges in the matching [20].

In a draft of this paper, we asked whether there exist FPT extended formulations for independent set (parameterized by solution size). Gajarskỳ et al. [17] answer this in the negative for arbitrary graphs, but provide FPT extended formulations for graphs of bounded expansion.

The independent set and vertex cover problems have both been studied from the perspective of approximate extended formulations. Independent set admits no polynomial-size *uniform* extended formulation that achieves an $O(n^{1-\epsilon})$ approximation for any constant $\epsilon > 0$ [9], which matches the inapproximability of the maximum independent set problem [18, 35]. If we allow for *non-uniform* extended formulations, that is, the inequalities defining the feasible region need not be the same for every $n$-vertex graph, then $O(1)$-approximate formulations still require superpolynomial size [4]. Somewhat surprisingly, $O(n^{1/2})$-approximate extended formulations of size $O(n)$ exist, but they are NP-hard to construct [4]. For the vertex cover

problem, there are no polynomial-size extended formulations achieving a $(2 - \epsilon)$-approximation [4] for any constant $\epsilon > 0$, and the standard linear programming relaxation provides a matching upper bound of 2.

Sometimes it will be convenient to work with the $k$-independent set polytope $\text{STAB}_k(G)$. We note that the independent set polytope $\text{STAB}(G)$ and vertex cover polytope $\text{VC}(G)$ have the same extension complexity. This can be seen by complementing the variables, i.e., replacing each instance of a variable $x_i$ in a formulation by $1 - x_i$. The same idea shows that their cardinality-constrained counterparts $\text{VC}_k(G)$ and $\text{STAB}_{n-k}(G)$ have the same extension complexity.

Other parameters besides solution size can be used when developing extended formulations for vertex cover. A notable example is the graph invariant treewidth $\mathbf{tw}$, which is a measure of how "tree-like" a graph is. There exist extended formulations for $\text{VC}(G)$ of size $O(2^{\mathbf{tw}}n)$ [23] (cf. an alternative proof by Buchanan and Butenko [10] based on the framework of Martin et al. [24]). Similar results hold for more general problems [5, 21, 22].

## 2. Formulations for Vertex Cover in Hypergraphs

The main result of this section is that there are simple size $O(d^k n)$ extended formulations for $k$-vertex covers of $d$-uniform hypergraphs. This bound is $O(n)$ when $d+k$ is a constant. The extended formulations are based primarily on the following foklore lemma, which we prove for completeness.

**Lemma 1** (folklore)**.** *In a $d$-uniform hypergraph $H = (V, E)$, the number of minimal vertex covers of $H$ that have size $\leq k$ is at most $d^k$.*

*Proof.* Denote by $S_k(H)$ the set of minimal vertex covers that have size at most $k$. We are to show that $|S_k(H)| \leq d^k$. The proof is by induction on $k$. In the base case, $k = 0$. If $E = \emptyset$, then $S_0(H) = \{\emptyset\}$ so $|S_0(H)| = 1 \leq d^0$. If not, then $E \neq \emptyset$ so $S_0(H) = \emptyset$ and $|S_0(H)| = 0 \leq d^0$. Now suppose the claim holds for $k = p$. If $E = \emptyset$, then $|S_{p+1}(H)| = 1$. If not, let $e \in E$ so $|S_{p+1}(H)| \leq \sum_{j \in e} |S_p(H - j)| \leq d \cdot d^p$. $\qquad\square$

The bound in Lemma 1 is sharp. It is achieved on the $d$-uniform hypergraph consisting of $k$ disjoint hyperedges.

**Proposition 1.** *If $H$ is a $d$-uniform $n$-vertex hypergraph, then the $k$-vertex cover polytope $\text{VC}_k(H)$ of $H$ admits an extended formulation of size $O(d^k n)$.*

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*Proof.* Define $S_k(H)$ as in the proof of Lemma 1. For each $S \in S_k(H)$, define $P(S) := \{x \in [0,1]^n \mid \sum_{i \in V} x_i = k;\ x \geq x^S\}$, where $x^S$ denotes the characteristic vector of $S$. We claim $\text{VC}_k(H) = \text{conv.hull}\left(\cup_{S \in S_k(H)} P(S)\right) =: P$, in which case the proposition would follow by Theorem 1.

($\subseteq$) Consider an extreme point of $\text{VC}_k(H)$, which will be $x^C$ for some $k$-vertex cover $C \subseteq V$. Then, $C$ is a superset of some minimal vertex cover $C'$, and $C' \in S_k(H)$. Thus, $x^C \in P(C') \subseteq P$.

($\supseteq$) Let $x'$ be an extreme point of $P$, which will be an extreme point of some $P(S)$. It is straightforward to show that $P(S)$ is an integer polytope, so $x' = x^D$ for some $D \subseteq V$. Thus $D \supseteq S$ and $|D| = k$, meaning that $D$ is a $k$-vertex cover so $x' = x^D$ belongs to $\text{VC}_k(H)$. $\qquad\square$

We remark that a better size bound $O(f(d,k) + kn)$ can be achieved, where $f$ is a function that depends only on $d$ and $k$. To achieve this, one can exploit the "full kernel" given in Theorem 7 of Damaschke's paper [15]. We employ the full kernel in the next section, and thus do not spend time detailing the approach for hypergraphs.

## 3. Improved FPT Formulations for Vertex Cover

The results of the previous section immediately imply size $O(2^k n)$ extended formulations for the $k$-vertex cover polytopes of $n$-vertex graphs. In this section, we improve this bound in two ways. One improvement is to reduce the base of the exponential term from 2 to 1.47 using a decomposition theorem of Chen et al. [12]. The other improvement is to use the "full kernel" of Damaschke [15] to separate the exponential term from the quantity $n$, resulting in a bound of the form $O(f(k) + kn)$.

Our first lemma comes from the full kernel given by Damaschke [15], which is itself a refinement of the kernel given by Sam Buss [11].

**Lemma 2** (via Theorem 3 of [15]). *If a graph $G = (V, E)$ has a $k$-vertex cover, then there is a subset $Z_k(G) \subseteq V$ of vertices such that*

$$|V| - |Z_k(G)| \leq \frac{1}{4}(k+1)^2 + k$$

*and no vertex of $Z_k(G)$ belongs to a minimal vertex cover of size at most $k$.*

The full kernel $V \setminus Z_k(G)$ can be found in time $O(m + T_k)$, where $T_k$ is the time to find a $k$-vertex cover in $G$ or determine that none exist [15]. Chen et al. [13] show that $T_k = O(1.2738^k + kn)$.

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We use the notation $x|_S$. Here, $x$ is a $|V|$-dimensional vector, $S \subseteq V$ is a subset of vertices, and $x|_S := [x_i]_{i \in S}$ is an $|S|$-dimensional vector. Also, $N(u) := \{v \in V \mid \{u, v\} \in E\}$ denotes the neighborhood of vertex $u$.

**Lemma 3.** *If $G = (V, E)$ is an $n$-vertex graph and $k$ is a positive integer, then*

$$\mathrm{VC}_k(G) = \mathrm{conv.hull}\left(\cup_{j=0}^k P_j\right) =: P.$$

*Here, $P_j$ is defined to be empty if $G$ has no $j$-vertex cover. Otherwise, consider a $Z_j(G)$ satisfying Lemma 2, $G_j := G - Z_j(G)$, and $P_j$ is the set of all $x \in [0,1]^n$ satisfying*

$$\sum_{i \in Z_j(G)} x_i = k - j \tag{1}$$

$$x|_{V \setminus Z_j(G)} \in \mathrm{VC}_j(G_j) \tag{2}$$

$$x_v = 1 \qquad\qquad v \in N(z), \ z \in Z_j(G). \tag{3}$$

*Hence, $\mathrm{xc}(\mathrm{VC}_k(G)) \leq (k+1)(2n+1) + \sum_{j=0}^k \mathrm{xc}(\mathrm{VC}_j(G_j))$.*

*Proof.* The polytope $Q_j$ defined as the set of all $x \in [0,1]^n$ satisfying constraints (1) and (2) has extension complexity at most $2n + \mathrm{xc}(\mathrm{VC}_j(G_j))$. Since $P_j$ is a face of $Q_j$, we have $\mathrm{xc}(P_j) \leq \mathrm{xc}(Q_j)$. Thus, $P$ has extension complexity at most $(k+1) + \sum_{j=0}^k (2n + \mathrm{xc}(\mathrm{VC}_j(G_j)))$ by Theorem 1. To complete the proof, we show $\mathrm{VC}_k(G) = P$. If either is empty, so is the other.

($\supseteq$) Let $x'$ be an extreme point of $P$, which will be an extreme point of some $P_j$. It can be argued that $\mathrm{proj}_{x|_{Z_j(G)}} Q_j$ is the convex hull of 0-1 vectors of dimension $|Z_j(G)|$ that have $k - j$ ones. Thus, $Q_j$ is integral, and so is its face $P_j$. So, $x' = x^D$ for some $D \subseteq V$. Edges of $G$ having one endpoint in $Z_j(G)$ are covered by $D$ by constraint (3), and no edge has both endpoints in $Z_j(G)$. Edges with both endpoints in $V \setminus Z_j(G)$ are covered by constraint (2). Further, $|D| = |D \cap Z_j(G)| + |D \cap (V \setminus Z_j(G))| = (k - j) + j = k$, so $D$ is a $k$-vertex cover for $G$, i.e., $x' \in \mathrm{VC}_k(G)$.

($\subseteq$) Consider an extreme point of $\mathrm{VC}_k(H)$, which will be $x^C$ for some $k$-vertex cover $C \subseteq V$. Then, $C$ is a superset of some minimal vertex cover $C'$. Let $q = |C'|$. Then $C' \subseteq V \setminus Z_q(G)$ and $x^C \in P_q \subseteq P$. $\qquad\square$

In the following theorem, we say that a vertex cover $C$ of $G$ is *consistent* with a partition $(F, D, R)$ of the vertices of a graph $G$ if $F \subseteq C$ and $D \cap C = \emptyset$. The idea is that vertices from $F$ are fixed in the cover, vertices from $D$ are fixed out of the cover, and the remaining vertices from $R$ are undetermined.

**Theorem 2** (Chen et al. [12]). *For every graph $G = (V, E)$ and every positive integer $k$, there is a collection $\mathcal{L}(G, k)$ of triples satisfying:*

1. $|\mathcal{L}(G, k)| \leq 1.466^k$;
2. *each $(F, D, R) \in \mathcal{L}(G, k)$ is a partition of $V$;*
3. *each $k$-vertex cover of $G$ is consistent with exactly one triple in $\mathcal{L}(G, k)$;*
4. *for each $(F, D, R) \in \mathcal{L}(G, k)$, the maximum degree of $G[R]$ is $\leq 2$.*

Chen et al. [12] provide an algorithm, running in time $O(1.47^k n)$, that finds such a collection $\mathcal{L}(G, k)$.

Theorem 2 decomposes the $k$-vertex covers of the graph into at most $1.466^k$ "easy" pieces, and this will allow us to decompose the $k$-vertex cover polytope similarly. Much of what remains is to show how to handle the easy pieces, i.e., that there is a small polyhedral description for cardinality-constrained vertex covers in the graph $G[R]$ in Theorem 2. First, we identify a small formulation for the $k$-independent set polytope of such a graph. This will imply similar results for cardinality-constrained vertex covers.

We will use the following theorem. It follows from known results about the matching polytope [30] as noted by Walter and Kaibel [33] (see also [32]).

**Theorem 3.** *The intersection of the matching polytope of a graph with a cardinality constraint is integral.*

Recall that the line graph $L(G)$ of a graph $G$ has vertex set $E(G)$ and two vertices from $L(G)$ are adjacent if the corresponding edges from $G$ are incident. A graph $H$ is said to be a line graph if there exists a graph $G$ such that $H = L(G)$. By the equivalence of the matching polytope of $G$ and the independent set polytope of its line graph $L(G)$, we have the following.

**Corollary 1.** *If $G = (V, E)$ is a line graph and $k$ is an integer, then*

$$\mathrm{STAB}_k(G) = \mathrm{STAB}(G) \cap \left\{ x \ \middle| \ \sum_{i \in V} x_i = k \right\}.$$

Corollary 1 does not hold for arbitrary graphs; it fails for the claw

$$K_{1,3} = ([4], \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}).$$

Indeed, $\hat{x} = \frac{1}{2}(1, 0, 0, 0) + \frac{1}{2}(0, 1, 1, 1)$ belongs to $\mathrm{STAB}(K_{1,3})$ and $\sum_{i=1}^{4} \hat{x}_i = 2$, but $\hat{x}$ violates the constraint $x_1 = 0$, which is valid for $\mathrm{STAB}_2(K_{1,3})$.

**Lemma 4.** *If $G = (V, E)$ is an $n$-vertex graph of maximum degree at most two, then, for any integer $p$, $\mathrm{STAB}_p(G)$ is the set of $x \in [0,1]^n$ satisfying:*

$$\sum_{i \in V} x_i = p \tag{4}$$

$$x_i + x_j \leq 1, \ \forall \{i, j\} \in E \tag{5}$$

$$\sum_{i \in C} x_i \leq (|C| - 1)/2, \ \textit{for every odd cycle } C. \tag{6}$$

*Hence, $\mathrm{xc}(\mathrm{VC}_k(G)) \leq \frac{11}{5}n$ for graphs $G$ of maximum degree at most two.*

*Proof.* Recall that a graph is said to be $t$-perfect if its independent set polytope is fully described by the 0-1 bounds, edge inequalities (5), and odd-cycle inequalities (6). Since $G$ has maximum degree at most two, it is the disjoint union of cycle and path graphs and is thus series-parallel. Series-parallel graphs are $t$-perfect, as first shown by [6] (cf. Chapter 68 of [30]). Thus, $\mathrm{STAB}(G) = \{x \in [0,1]^n \mid x \text{ satisfies (5) and (6)}\}$, and since $G$ is a line graph, the first claim follows by Corollary 1. To see a size $\frac{11}{5}n$ bound for $\mathrm{STAB}_p(G)$, it is enough to show that for each component $G'$ there are at most $\frac{11}{5}|V(G')|$ irredundant corresponding inequalities. This is achieved by the 5-cycle, and it can be seen that no other component has more. The last claim holds by letting $p = n - k$ and complementing the variables. $\qquad\square$

**Lemma 5.** *The $k$-vertex cover polytope of an $n$-vertex graph $G$ has extension complexity at most $1.466^k \left( \frac{11}{5}n + 1 \right)$.*

*Proof.* By Theorem 2, we can write $\mathrm{VC}_k(G)$ as follows.

$$\mathrm{VC}_k(G) = \mathrm{conv.hull} \left( \bigcup_{(F,D,R)} P_k(F, D, R) \right).$$

Here, the union is taken over all $(F, D, R) \in \mathcal{L}(G, k)$ and $P_k(F, D, R)$ is the convex hull of $k$-vertex covers that are consistent with $(F, D, R)$. We assume that no edge $\{u, v\}$ of $G$ has $u \in D$ and $v \in R$, for if this is the case then we can move $v$ to $F$. The polytope $P_k(F, D, R)$ can thus be written as the set of all $x \in \mathbb{R}^n$ satisfying $x_i = 1$ for all $i \in F$, $x_i = 0$ for all $i \in D$, and $x|_R \in \mathrm{VC}_{k-|F|}(G[R])$. By Lemma 4, the constraints for $x|_R$ can be written using no more than $\frac{11}{5}|R| \leq \frac{11}{5}n$ inequalities. The claim follows by Theorem 1. $\quad\square$

**Theorem 4.** *The $k$-vertex cover polytope of an $n$-vertex graph $G$ admits an extended formulation of size $O(1.47^k + kn)$.*

*Proof.* Define $G_j$ as in Lemma 3. Then,

$$\text{xc}(\text{VC}_k(G)) \leq (k+1)(2n+1) + \sum_{j=0}^{k} \text{xc}(\text{VC}_j(G_j)) \tag{7}$$

$$\leq (k+1)(2n+1) + \sum_{j=0}^{k} 1.466^j \left( \frac{11}{5}|V(G_j)| + 1 \right) \tag{8}$$

$$\leq (k+1)(2n+1) + \sum_{j=0}^{k} 1.466^j \left( \frac{11}{5} \left( \frac{1}{4}(j+1)^2 + j \right) + 1 \right) \tag{9}$$

$$= O\left( kn + 1.466^k k^3 \right) \tag{10}$$

$$= O\left( kn + 1.47^k \right). \tag{11}$$

Here, inequality (7) holds by Lemma 3, inequality (8) holds by Lemma 5, and inequality (9) holds by Lemma 2. □

## 4. Conclusion

We provide FPT extended formulations for the vertex cover polytopes of (hyper)graphs, with the size $k$ of the cover as the parameter. An initial extended formulation has size $O(d^k n)$ for the case of $d$-uniform hypergraphs, which implies size $O(2^k n)$ extended formulations for graphs. Using structural results from the FPT literature and known polyhedral results about matching and independent set polyhedra, we are able to improve the bound to $O(1.47^k + kn)$.

It is likely that smaller extended formulations exist. One approach is to use more insights from the FPT literature. For example, one algorithm solves the (unweighted) $k$-vertex cover problem in time $O(1.2738^k + kn)$ [13]. However, it relies on reduction rules that remove feasible solutions, so its ideas may not be directly applicable when developing extended formulations. For the weighted variant, an algorithm of Chen et al. [12] finds the $K$ cheapest $k$-vertex covers in time $O(1.47^k n + 1.22^k Kn)$. We are aware of no faster, published algorithm to find a cheapest $k$-vertex cover, although this would not surprise us if one existed.

It would be interesting to find matching lower and upper bounds for the extension complexity of the $k$-vertex cover polytope. For example, can we rule out extended formulations of size $2^{o(k)} n^{O(1)}$? A weaker claim holds.

**Remark 1.** *The k-vertex cover polytopes of n-vertex graphs do not admit extended formulations of size $2^{o(\sqrt{k})}n^{O(1)}$.*

*Proof.* For any graph $G$, $\text{VC}(G) = \text{conv.hull}\left\{\cup_{k=0}^{n}\text{VC}_k(G)\right\}$. So, by Theorem 1, if $\text{xc}(\text{VC}_k(G)) = 2^{o(\sqrt{k})}n^{O(1)}$, then $\text{xc}(\text{STAB}(G)) = \text{xc}(\text{VC}(G)) = 2^{o(\sqrt{n})}$, which contradicts that independent set polytopes of some graphs have extension complexity $2^{\Omega(\sqrt{n})}$ [16]. $\square$

It has been conjectured [7] that the $2^{\Omega(\sqrt{n})}$ bound for independent set polytopes can be strengthened to $2^{\Omega(n)}$. If this is true, then there are no size $2^{o(k)}n^{O(1)}$ extended formulations for $k$-vertex cover, by the same argument.

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