Extended formulations for vertex cover[1]

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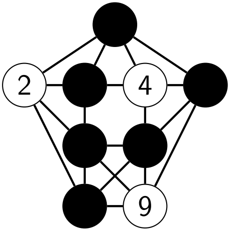
Outline

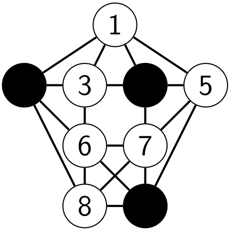
Section 1

The vertex cover polytope

In graph $G = (V, E)$, a vertex cover is a vertex subset that touches each edge at least once:



A stable set touches each edge at most once:



Observation: $S \subseteq V$ is vertex cover $\iff$ $V \setminus S$ is stable set.

We can represent the collection of all vertex covers polyhedrally:

The vertex cover polytope of a graph $G = (V, E)$ is denoted

$$\begin{aligned} \mathrm{VC}(G) &:= \text{conv.hull}\left\{x^S \;\middle|\; S \subseteq V \text{ is a vertex cover for } G\right\} \\ &= \text{conv.hull}\left\{x \in \{0,1\}^n \mid x_i + x_j \geq 1,\ \forall \{i,j\} \in E\right\}. \end{aligned}$$

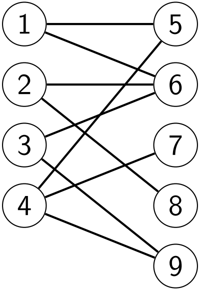
Similarly, the stable set polytope of G is

$$\begin{aligned} \mathrm{STAB}(G) := &\ \text{conv.hull}\left\{x^S \;\middle|\; S \subseteq V \text{ is a stable set in } G\right\} \\ = &\ \text{conv.hull}\left\{x \in \{0,1\}^n \mid x_i + x_j \leq 1,\ \forall \{i,j\} \in E\right\}. \end{aligned}$$

$\mathrm{STAB}(G)$ is well-studied in the literature, particularly to help solve problems with set packing constraints (Padberg, 1973).

Observation: $\mathrm{VC}(G) = \{x \mid (\mathbf{1} - x) \in \mathrm{STAB}(G)\}$.

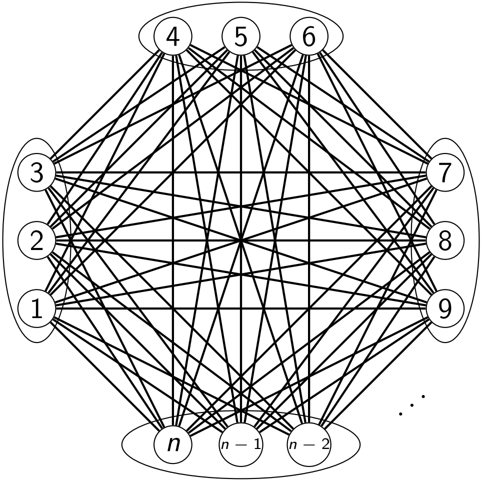
How complex are VC(G) and STAB(G)? (in $\mathcal{H}$-representation)



For bipartite graphs G, small integral formulations are easy:

$$\begin{aligned} \mathrm{VC}(G) &= \text{conv.hull}\,\{x \in \{0,1\}^n \mid x_i + x_j \geq 1,\ \forall \{i,j\} \in E\} \\ &= \{x \in [0,1]^n \mid x_i + x_j \geq 1,\ \forall \{i,j\} \in E\}\,. \end{aligned}$$

But, this isn't always the case in original variables:



$3^{n/3}$ facet-defining inequalities of type $\sum_{i \in C} x_i \geq \frac{n}{3} - 1$.

Small "perfect" formulations may exist using additional variables:

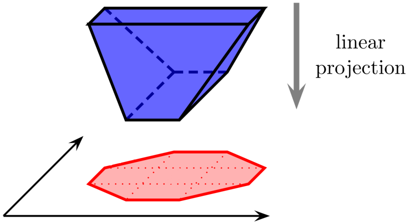


Figure: A smaller extension (image courtesy of Rothvoss)

Definition (Extension)

Let $P = \{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$ be a polyhedron. A polyhedron $Q \subseteq \mathbb{R}^{n+d}$ is said to be an extension for P if $\text{proj}_x(Q) = P$.

- size of an extension
- extended formulation

How few inequalities are needed?

Definition

The *extension complexity* of a polyhedron P is

$$\mathrm{xc}(P) := \min\{\mathrm{size}(Q) \mid Q \text{ is an extension for } P\}.$$

Do small extended formulations always exist?

Theorem (Fiorini et al. (2012))

There is a class of graphs G for which $\mathrm{xc}(\mathrm{VC}(G)) = 2^{\Omega(\sqrt{n})}$.

Theorem (Göös et al. (2016))

There is a class of graphs G for which $\mathrm{xc}(\mathrm{VC}(G)) = 2^{\Omega(n/\log n)}$.

Theorem (Bazzi et al. (2015))

No polysize LPs that approximate $\mathrm{VC}(G)$ *within a factor* $2 - \varepsilon$.

We take the perspective of parameterized complexity:

- How does extension complexity depend on solution size k?

The k-vertex cover polytope:

$$\begin{aligned}\mathrm{VC}_k(G) &:= \text{conv.hull}\left\{x^S \;\middle|\; S \subseteq V \text{ is a } k\text{-vertex cover for } G\right\}\\ &= \text{conv.hull}\left\{x \in \{0,1\}^n \mid x(V) = k;\ x_i + x_j \geq 1,\ \forall \{i,j\} \in E\right\}\end{aligned}$$

Obviously, there are size $O(n^k)$ extended formulations:

$$
\begin{aligned}
x_i &= \sum_{S: i \in S} y_S & i \in V \\
\sum_S y_S &= 1 \\
y_S &\geq 0 & \text{for each } k\text{-vertex cover } S \subseteq V.
\end{aligned}
$$

But, can you achieve fixed-parameter tractability, i.e., of the form

$$f(k)n^{O(1)}$$

for some function f?

Our main results

In a d-uniform hypergraph $H = (V, E)$, we have $E \subseteq \binom{V}{d}$.

Proposition

There are size $O(d^k n)$ extended formulations for the k-vertex cover polytopes of n-vertex d-uniform hypergraphs.

Corollary: size $O(2^k n)$ for graphs.

Theorem

There are size $O(1.47^k + kn)$ extended formulations for the k-vertex cover polytopes of n-vertex graphs.

Section 2

Preliminary extended formulation (for hypergraphs)

It is easy to find an $\mathcal{H}$-representation for the intersection

$$P^{\cap} := \bigcap_{i=1}^{q} P^i$$

of $\mathcal{H}$-polyhedra $P^1, \dots, P^q$. In fact, $\mathrm{xc}(P^{\cap}) \leq \sum_{i=1}^{q} \mathrm{xc}(P^i)$.

Taking the union $P^{\cup} := \text{conv.hull}\left(\bigcup_{i=1}^{q} P^i\right)$ is almost as easy:

Theorem (Balas)

$$\mathrm{xc}(P^{\cup}) \leq q + \sum_{i=1}^{q} \mathrm{xc}(P^i).$$

Proposition

There are size $O(d^k n)$ extended formulations for the k-vertex cover polytopes of n-vertex d-uniform hypergraphs.

Proof. Focus on minimal vertex covers S with $|S| \leq k$. An integral formulation for the k-vertex covers that are supersets of S is:

$$P(S) := \left\{ x \in [0,1]^n \;\middle|\; \sum_{i \in V} x_i = k;\ x \geq x^S \right\}.$$

Denote by $S_k(H)$ the family of all such S.It is easy to argue:

$$\mathrm{VC}_k(H) = \text{conv.hull}\left(\bigcup_{S \in S_k(H)} P(S) \right).$$

Key folklore result: $|S_k(H)| \leq d^k$. So, by Balas's theorem,

$$\mathrm{xc}(\mathrm{VC}_k(H)) \leq |S_k(H)| + \sum_{S \in S_k(H)} \mathrm{xc}(P(S)) \leq d^k + d^k * 2n = O(d^k n).$$

Section 3

Improved extended formulation (for graphs)

The previous proposition implies:

Corollary

There are size $O(2^k n)$ extended formulations for the k-vertex cover polytopes of n-vertex graphs.

Now, we improve this in two ways:

1. reduce the base of the exponential term from 2 to 1.47;
2. separate the exponential term from n, yielding size $O(f(k) + kn)$.

We can break up the k-vertex covers into $\leq 1.466^k$ easy pieces and write a small formulation for each easy piece, based on:

Theorem (decomposition theorem of Chen et al. (2013))

For every graph $G = (V, E)$ and every positive integer k, there is a collection $\mathcal{L}(G, k)$ of triples satisfying:

1. $|\mathcal{L}(G, k)| \leq 1.466^k$;
2. *each* $(F, D, R) \in \mathcal{L}(G, k)$ *is a partition of* V;
3. *each* k*-vertex cover is consistent with one triple in* $\mathcal{L}(G, k)$;
4. $\forall (F, D, R) \in \mathcal{L}(G, k)$, $G[R]$ *is paths + cycles.*

- $C \subseteq V$ is consistent with (F, D, R) if $F \subseteq C$ and $D \cap C = \emptyset$.
- WLOG, we assume no edge $\{u, v\}$ with $u \in D$, $v \in R$.

Lemma

The k-vertex cover polytope of an n-vertex graph G has extension complexity at most $1.466^k\left(\frac{11}{5}n+1\right)$.

By Chen et al.'s decomposition theorem, we can write $\mathrm{VC}_k(G)$ as:

$$\mathrm{VC}_k(G) = \text{conv.hull}\left(\bigcup_{(F,D,R)\in\mathcal{L}(G,k)} P_k(F,D,R)\right),$$

where

$$P_k(F,D,R) := \text{conv.hull}\left\{x^C \in \mathrm{VC}_k(G) \;\middle|\; F \subseteq C,\ D \cap C = \emptyset\right\}$$

is the convex hull of k-vertex covers consistent with (F,D,R).

By Balas, we just need size $\frac{11}{5}n$ formulations for each $P_k(F,D,R)$.

How to formulate $P_k(F, D, R)$ using at most $\frac{11}{5}n$ inequalities?

1. $x_i = 1$ for $i \in F$;
2. $x_i = 0$ for $i \in D$;
3. $x|_R \in \mathrm{VC}_{k-|F|}(G[R])$.

Need to exploit properties of $G[R]$.

Recall: $G[R]$ is the disjoint union of paths and cycles.
$\implies$ $G[R]$ is series-parallel and a line graph.

Theorem (Boulala and Uhry (1979))

Series-parallel graphs G are t-perfect, i.e., STAB(G) *is* $x \in [0,1]^n$ *:*

$$\begin{aligned} x_i + x_j &\leq 1 && \text{for every } \{i,j\} \in E; \\ \sum_{i \in C} x_i &\leq (|C| - 1)/2 && \text{for every odd cycle } C \subseteq V. \end{aligned}$$

By known results about matching polyhedra:

Lemma (Folklore, cf. Walter and Kaibel (2015))

If $G = (V, E)$ is a line graph and p is an integer, then

$$\mathrm{STAB}_p(G) = \mathrm{STAB}(G) \cap \left\{ x \;\middle|\; \sum_{i \in V} x_i = p \right\}.$$

Thus, $\mathrm{STAB}_p(G[R])$ is the set of all $x|_R \in [0,1]^{|R|}$ satisfying:

$$
\begin{aligned}
&\sum_{i \in R} x_i = p \\
&x_i + x_j \leq 1 && \text{for every } \{i,j\} \in E(G[R]) \\
&\sum_{i \in C} x_i \leq (|C| - 1)/2 && \text{for every odd cycle } C \subseteq R.
\end{aligned}
$$

Observation: there are at most $\frac{11}{5}|R|$ irredundant inequalities.

Corollary

$\mathrm{xc}(\mathrm{STAB}_p(G[R])) \leq \frac{11}{5}|R|$.

Setting $p := n - k$ and complementing variables gives:

Corollary

$\text{xc}(\text{VC}_k(G[R])) \leq \frac{11}{5}|R|.$

This finally shows our lemma: $\text{xc}(\text{VC}_k(G)) \leq 1.466^k(\frac{11}{5}n + 1).$

Theorem

There are size $O(1.47^k + kn)$ *extended formulations for the* k*-vertex cover polytopes of* n*-vertex graphs.*

This follows (by simple arguments) through kernelization:

Theorem ("Full kernel" of Damaschke (2006))

If a graph $G = (V, E)$ *has a* k*-vertex cover, then there is a subset* $Z_k(G) \subseteq V$ *of vertices such that*

$$|V| - |Z_k(G)| \leq \frac{1}{4}(k+1)^2 + k$$

and no vertex of $Z_k(G)$ *belongs to a minimal VC of size* $\leq k$*.*

Section 4

Conclusion

Our contributions

We provide extended formulations of size:

1. $O(d^k n)$ for k-vertex covers in d-uniform hypergraphs;
2. $O(1.47^k + kn)$ for k-vertex covers in graphs.

Key proof ingredients:

- From parameterized complexity:
  1. Damaschke's full kernel for k-vertex cover
  2. Chen et al.'s decomposition theorem for k-vertex cover
- From polyhedral combinatorics:
  1. Balas's extended formulation for union of polyhedra
  2. t-perfect graphs
  3. cardinality-constrained matching polyhedra

Related Work and Future Directions

What about the extension complexity of $\mathrm{STAB}_k(G)$?

Remark (follows from Fiorini et al. (2012))

There are no EFs of size $2^{o(\sqrt{k})}n^{O(1)}$.

Theorem (Gajarskỳ et al. (2015))

There are no EFs of size $f(k)n^{O(1)}$.

Open question. Are there EFs of size $2^{o(k)}n^{O(1)}$ for k-vertex cover? In other words, are our EFs essentially optimal?

Thanks!

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