On connected dominating sets of restricted diameter

Austin Buchanan, Je Sang Sung, Vladimir Boginski, Sergiy Butenko

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Abstract

A connected dominating set (CDS) is commonly used to model a virtual backbone of a wireless network. To bound the distance that information must travel through the network, we explicitly restrict the diameter of a CDS to be no more than s leading to the concept of a dominating s-club. We prove that for any fixed positive integer s it is NP-complete to determine if a graph has a dominating s-club, even when the graph has diameter $s+1$. As a special case it is NP-complete to determine if a graph of diameter two has a dominating clique. We then propose a compact integer programming formulation for the related minimization problem, enhance the approach with variable fixing rules and valid inequalities, and present computational results.

keywords: connected dominating set, bounded diameter subgraphs, s-club, clique, wireless networks

1 Introduction

It is common to represent a wireless communication network as a graph, where nodes in the wireless network correspond to vertices in the graph and edges denote the ability to communicate directly. Any given pair of nodes may not be close enough to communicate directly, but a message can be transmitted through intermediate nodes. In the wireless network, these intermediate nodes form a virtual backbone structure, which is a connected dominating set (CDS) in the graph setting. It may be important to transmit these messages quickly. This requirement of low-latency can be enforced in the graph by requiring that the vertices from the CDS induce a subgraph of small diameter. This leads to the notion of a dominating s-club—a dominating set whose induced subgraph has diameter at most s. It can be seen that dominating s-club generalizes dominating clique (let $s = 1$), and if we allow s to be instance-dependent it generalizes CDS (let $s = n - 1$ for an n-vertex graph).

A dominating clique (i.e., a dominating 1-club) introduced by [8] (see also [9, 12]) can provide a virtual backbone structure for a wireless communication network that has the shortest communication time. Since the members of the dominating clique can communicate directly, a message departing from any source can reach any destination in at most 3 hops. On the other hand, ensuring the existence of a dominating clique in a wireless network may be too costly. Moreover, this may result in excessive interference.

A dominating s-club provides a compromise, possibly offering advantages over the commonly used CDS in terms of speed of communication, energy consumption, and reliability [22]. Small diameter dominating sets facilitate quick communication between each pair of vertices, as any two vertices are at most $s + 2$ hops apart: 1 hop to reach the backbone, s hops within the backbone, and 1 hop to the destination. All else being equal, smaller dominating sets contribute to improved energy consumption, as the number of transmissions required to pass a message from one vertex to another is small. Also recognize that long transmission paths can increase the chance of message transmission failure [27], meaning that a smaller-diameter backbone may be more reliable. In contrast to a dominating s-club, a CDS D ensures little in terms of the diameter (the worst case diameter is $|D|-1$), possibly resulting in excessively long transmission paths between vertices.

The minimum dominating s-club problem, which is the focus of the present paper, is defined as follows. Given a graph $G=(V,E)$ and a positive integer constant s, find a smallest dominating set $D \subseteq V$ such that $diam(G[D]) \leq s$, or decide that none exist. When a dominating s-club exists in G, we call the size of a minimum dominating s-club the *dominating s-club number* of G and denote it by $\gamma^s_{club}(G)$.

1.1 Notation and related work

We consider a simple undirected graph $G=(V,E)$ with set V of n vertices and set $E \subset V \times V$ of edges. We denote the (open) neighborhood of a vertex $i \in V$ by $N(i)=\{j \in V : \{i,j\} \in E\}$, and the closed neighborhood of $i \in V$ by $N[i]=N(i)\cup\{i\}$. A set $D \subseteq V$ is called a dominating set if each vertex in $V \setminus D$ has a neighbor in D, and a total dominating set if each vertex in V (including the vertices from D) has a neighbor in D. Let $G[S]$ be the graph induced by $S \subseteq V$. A dominating set that induces a connected graph is called a connected dominating set. Let the distance $d_G(i,j)$ be the length of a shortest path between vertices $i,j \in V$ in a graph G, and let $diam(G)=\max\{d_G(i,j) : i,j \in V\}$ be the diameter of G. We adopt the convention that $diam(G)=\infty$ for a disconnected graph G. A clique $C \subseteq V$ is a subset of pairwise adjacent vertices, i.e., $diam(G[C])=1$. [28] introduced a clique relaxation model called an *s-club*, which is a subset $S \subseteq V$ of vertices such that $diam(G[S]) \leq s$. An s-club that forms a dominating set is called a dominating s-club.

The minimum dominating clique problem has been shown to be polynomial-time solvable in strongly chordal graphs [23], undirected path graphs [23], and circle graphs [21]. In general, however, the minimum dominating clique problem is NP-hard [24]. In fact, it is NP-complete to determine the existence of a dominating clique in a graph [8, 9]. An exact, exponential-time algorithm has been developed for arbitrary graphs [24].

Since we want to restrict the diameter, the notion of s-club is important. Several papers in literature consider a different problem: the maximum s-club problem. Even though this problem is different from ours, it provides useful insights for complexity and IP formulations. The maximum s-club problem asks for a maximum size subset of vertices whose induced subgraph has diameter at most s. The decision version of the maximum s-club problem has been shown to be NP-complete [7], even when restricted to graphs of diameter $s+1$ [6]. Some exact approaches for the maximum s-club problem are based on integer programming formulations with $O(n^{s+1})$ entities [7, 6]. More recently, [34] proposed a compact formulation having $O(sn^2)$ entities. This compact formulation is key to the development of the formulation proposed in this paper, which also has $O(sn^2)$ entities. The reader is referred to [29] for more information on s-clubs and other clique relaxation models.

The NP-hard minimum connected dominating set (MCDS) problem and the related maximum leaf spanning tree problem have received significant attention in the literature. A wide variety of approaches have been considered, including exact approaches [13, 26, 32, 16, 17, 15, 14], approximation algorithms [18, 25], and polynomial-time approximation schemes for unit disk graphs [20, 11] and unit ball graphs [36].

Considerably less work has been done for restricted-diameter dominating sets. Researchers have proposed heuristics [27] and approximation algorithms [22] with constant ratios for both CDS size and diameter (for disk graphs). [31] has shown that it is NP-complete to determine if a graph of diameter $s+2$ has a dominating s-club. To the best of our knowledge, there are no exact approaches for the minimum dominating s-club problem in previous literature (for general s).

1.2 Our contributions

In Section 2, we prove that it is NP-complete to determine if a graph of diameter two has a dominating clique. We extend this for any fixed positive integer s, showing that it is NP-complete to determine if a graph of diameter $s+1$ admits a dominating s-club. We consider the minimization problem in Section 3, showing that it is NP-hard to approximate within a logarithmic factor, even when a dominating s-club is known to exist. We also relate the dominating s-club number to other domination-related graph invariants, and demonstrate that restricting the diameter by just one unit can be very costly. In Section 4, we present a compact integer programming formulation for the minimum dominating s-club problem and develop associated valid inequalities and variable fixing rules. Section 5 discusses the potential applicability of the formulation to solving the classical MCDS problem. Section 6 reports the results of numerical experiments with the proposed formulation for the minimum dominating s-club problem on a set of random unit disk graphs and MCDS instances from literature. Finally, Section 7 concludes the paper.

2 Existence of a dominating s-club

In this section we consider the questions: (1) which graphs have a dominating s-club? and (2) what is the computational complexity of determining if a graph admits a dominating s-club?

It is clear that not every graph has a dominating s-club, as evidenced by the class of disconnected graphs. However, even when the graph is connected, a dominating s-club may not exist; the cycle on six vertices has no dominating 2-club. Some have attempted to find sufficient conditions for a dominating s-club to exist. [12] show that a dominating clique exists if the graph has no induced C_5 and no induced P_5. (P_t and C_t are the path and cycle on t vertices, respectively.) See also [4]. This has been extended for other values of s [5, 33, 3]. Namely, for $s = 1$ or $s = 2$, a dominating s-club will exist if the graph has no induced C_{s+4} and no induced P_{s+4}; for $s \geq 3$, a dominating s-club will exist if there is no induced P_{s+4} [33].

In the following proposition, we note some simple cases where a dominating s-club will or will not exist.

Proposition 1 *Consider a graph $G = (V, E)$.*

1. *For any $s \geq diam(G)$, there exists a dominating s-club in G.*
2. *If $diam(G) \geq 4$ then for any $s < diam(G) - 2$, there does not exist a dominating s-club in G.*

Proof. It is clear that V is a dominating s-club in the first case. To prove the second case, consider vertices $v, v' \in V$ such that $d_G(v, v') = diam(G)$. We claim that a dominating s-club must include $u \in N(v)$ and $u' \in N(v')$. (Otherwise, v (or v') must be dominated by itself, in which case v (or v') is isolated and the dominating set is not connected.) Thus, for any dominating set $S \subseteq V$, $d_{G[S]}(u, u') \geq d_G(u, u') \geq diam(G) - 2 > s$, implying that u and u' cannot belong to the same s-club. □

This leaves two values of s where it is not yet clear if a dominating s-club exists or not: $s = diam(G) - 1$ and $s = diam(G) - 2$. It turns out that these cases are NP-complete. The case $s = diam(G) - 2 = 1$ (i.e., Dominating Clique in diameter 3 graphs) has been shown by [9]. We and [31] independently show that the problem is NP-complete for *each* fixed positive integer s. We note however that the reduction used by [31] has $s = diam(G) - 2$. Our reduction allows us to make a different statement—that the NP-completeness holds when $s = diam(G) - 1$. First, we show that determining if a diameter two graph has a dominating clique is NP-complete. Then, using this fact, we describe NP-completeness reductions for even $s \geq 2$ and odd $s \geq 3$ (both are provided in the appendix).

Theorem 1 *The problem of determining if a graph of diameter two has a dominating clique is NP-complete.*

Proof. The reduction is from 3SAT (with exactly three literals per clause), where the instance Φ is defined by clauses $C_1 \wedge \cdots \wedge C_m$ and variables $x_1, \ldots, x_n$. For convenience, we make an assumption about Φ that does not affect the problem's complexity. We assume that Φ is not satisfied by fixing only $t \leq 3$ variables.

We construct a graph $G = (V, E)$ of diameter two that has a dominating clique if and only if Φ is satisfiable. We abuse notation, referring to a variable x_i (or its negation $\overline{x}_i$) and its associated vertex x_i ($\overline{x}_i$) from G in the same way. We construct the vertex set $V = X \cup \overline{X} \cup Q$ as follows.

$$X = \{x_i : 1 \leq i \leq n\}, \tag{1}$$
$$\overline{X} = \{\overline{x}_i : 1 \leq i \leq n\}, \tag{2}$$
$$Q = Q^1 \cup \cdots \cup Q^5, \tag{3}$$
$$Q^p = \{c_{pj} : 1 \leq j \leq m\} \text{ for } p = 1, \ldots, 5. \tag{4}$$

So, we have a vertex for each variable, one for each negated variable, and five vertices for each clause. The edge set $E = E_1 \cup \cdots \cup E_7$ is defined as follows.

$$E_1 = \{\{x_i, x_j\} : 1 \leq i < j \leq n\}, \tag{5}$$
$$E_2 = \{\{\overline{x}_i, \overline{x}_j\} : 1 \leq i < j \leq n\}, \tag{6}$$
$$E_3 = \{\{x_i, \overline{x}_j\} : i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\}, \tag{7}$$
$$E_4 = Q^1 \times Q^5, \tag{8}$$
$$E_5 = \bigcup_{p=1}^{4} Q^p \times Q^{p+1}, \tag{9}$$
$$E_6 = \bigcup_{p=1}^{5} \{\{x_i, c_{pj}\} : x_i \text{ appears in clause } C_j\}, \tag{10}$$
$$E_7 = \bigcup_{p=1}^{5} \{\{\overline{x}_i, c_{pj}\} : \overline{x}_i \text{ appears in clause } C_j\}. \tag{11}$$

The reduction is clearly polynomial in time and space, and it is easy to see that graph G has diameter two. Note that the edge subsets E_1, E_2, E_3 ensure that the edges from $G[X \cup \overline{X}]$ comprise a complete graph with a perfect matching $\{\{x_i, \overline{x}_i\} : i = 1, \ldots, n\}$ removed.

Claim 1: if the instance of 3SAT is satisfiable, then G has a dominating clique. Given a satisfying assignment x^*, we construct a dominating clique S of G. For $i = 1, \ldots, n$, choose vertex x_i if x_i^* is set to true and choose vertex $\overline{x}_i$ otherwise. This set S of n chosen vertices induces a complete subgraph (since no pair $\{x_i, \overline{x}_i\}$ will be simultaneously chosen), so S is a clique. It is easy to see that clause vertices from $Q^p, p = 1, \ldots, 5$ will be dominated by S due to the way in which the edge subsets E_6 and E_7 were created. Finally, every vertex from $X \cup \overline{X}$ will be dominated by S, so S is dominating.

Table 1: Complexity of DOMINATING s-CLUB for graphs G for any fixed $s \geq 1$.

value of s	complexity	comment
$s \leq diam(G) - 3$	P	answer is always 'no'
$s = diam(G) - 2$	NP-complete	by [31]
$s = diam(G) - 1$	NP-complete	by Theorem 2
$s \geq diam(G)$	P	answer is always 'yes'

Claim 2: no vertex from Q can belong to a dominating clique $S \subseteq V$ of G. See that for $p = 1, \dots, 5$ at most one vertex from Q^p can belong to a clique since Q^p is an independent set. We consider two cases where $|S \cap Q| \geq 1$. In each case, we arrive at a contradiction.

1. Case $|S \cap Q| \geq 2$. Without loss of generality, we can assume that $|S \cap Q^1| = |S \cap Q^2| = 1$. This implies that $S \cap Q^3 = S \cap Q^4 = S \cap Q^5 = \emptyset$ since no vertex in $Q^3 \cup Q^4 \cup Q^5$ has neighbors in both Q^1 and Q^2. Let $c_{1j} \in S \cap Q^1$ and $c_{2k} \in S \cap Q^2$. This set $\{c_{1j}, c_{2k}\}$ is not dominating on its own, since no vertex in Q^4 will be dominated. So, we must have $|S \cap (X \cup \overline{X})| \geq 1$. However, $|N(c_{1j}) \cap N(c_{2k})| \leq 3$, so $|N(c_{1j}) \cap N(c_{2k}) \cap S| \leq 3$. By the assumption about Φ that no three variables can satisfy Φ on their own, S cannot dominate the vertices from Q^4.

2. Case $|S \cap Q| = 1$. Without loss of generality, we can assume that $|S \cap Q^1| = 1$. The proof is very similar to that of the previous case and is omitted.

Claim 3: if G has a dominating clique, then the 3SAT instance is satisfiable. By Claim 3, a dominating clique of G can only use vertices from $X \cup \overline{X}$. Given a dominating clique $S \subset X \cup \overline{X}$, we construct a satisfying assignment x^*. For each vertex $x_i \in X$ do the following. If the vertex x_i belongs to S then set x_i^* to true; otherwise, set x_i^* to false. It is clear that since every clause vertex from Q^1 was dominated by S, the assignment x^* is satisfying. □

Theorem 2 *For any fixed positive integer s, the problem of determining if a graph of diameter $s+1$ has a dominating s-club is NP-complete.*

The previous theorem holds by Lemmata 1 and 2 from the appendix. Thus, by Theorem 2, Proposition 1, and [31], a sharp transition in the complexity of DOMINATING s-CLUB emerges, which is highlighted in Table 1.

3 The minimization problem

Here we consider the minimum dominating s-club problem, and relate the dominating s-club number $\gamma^s_{club}(G)$ to other domination-based graph invariants. Recall that $\gamma(G), \gamma_t(G), \gamma_c(G), \gamma_{cl}(G)$ denote the domination number, the total domination number, the connected domination number, and the dominating clique number of G, respectively. [19].

Remark 1 *If a graph G has a dominating clique and $\gamma(G) \geq 2$, then for $s' \geq s \geq 2$ the following inequalities hold:*

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G) \leq \gamma^{s'}_{club}(G) \leq \gamma^s_{club}(G) \leq \gamma_{cl}(G). \tag{12}$$

This remark shows that requiring a smaller diameter may come at the cost of a larger set. Given that a smaller diameter dominating set is usually preferred, it would be nice to bound this extra cost. Unfortunately, this cost can be very large indeed. This claim, stated formally in Remark 2, is demonstrated in the appendix.

Remark 2 *For any fixed positive integer s, the ratio $\gamma^s_{club}(G)/\gamma^{s+1}_{club}(G)$ for a graph G can be arbitrarily large.*

Now we consider the complexity of the minimum dominating s-club problem. Of course this is a difficult problem, because it is NP-complete to determine if *any* feasible solution exists. However, it could be that the difficulty lies only in finding a feasible solution. Remark 3, below, states that this is not the case. The minimization problem remains hard to approximate when a feasible solution is known.

Recall that the minimum set cover problem is NP-hard to approximate to within a factor of $c \log n$ for c constant [30, 2], where n refers to the number of elements that must be covered. [18] have shown that the MCDS problem is hard to approximate to within a factor of $c \log \Delta$, where Δ is the maximum degree of the graph, via a reduction from the minimum set cover problem. We note that the same reduction from [18] shows that the minimum dominating s-club problem is hard to approximate for any fixed $s \geq 2$, even when a feasible solution is known to exist. This holds because their construction ensures that $\gamma^2_{club}(G) = \gamma_c(G)$, and Remark 1 states that $\gamma^s_{club}(G)$ will be sandwiched between these values.

Remark 3 (Inapproximability) *Let $s \geq 2$ be a fixed integer. It is NP-hard to approximate the minimum dominating s-club problem to within a factor of $c \log \Delta$ where c is a constant (and Δ is the maximum degree), even when a dominating s-club is known to exist.*

4 IP formulation, valid inequalities, and variable fixing

We use the s-club (diameter) constraints proposed by [34] for the maximum s-club problem (constraints (15)–(20)). Let $V = \{1, 2, \ldots, n\}$ be the set of vertices. For $D \subseteq V$ the vector $x \in \{0,1\}^n$ such that $x_i = 1$ iff $i \in D$ is called the characteristic vector of D. Then the minimum dominating s-club problem can be formulated as the following IP.

$$\text{minimize} \quad \sum_{i \in V} x_i \tag{13}$$

$$\text{subject to} \quad \sum_{j \in N[i]} x_j \geq 1, \qquad i \in V; \tag{14}$$

$$\sum_{r=2}^{s} y^{(r)}_{ij} \geq x_i + x_j - 1, \qquad i \in V, j \in V \setminus N[i]; \tag{15}$$

$$y^{(2)}_{ij} \leq x_i, \quad y^{(2)}_{ij} \leq x_j, \quad y^{(2)}_{ij} \leq \sum_{p \in N(i) \cap N(j)} x_p, \qquad j > i, \ i, j \in V; \tag{16}$$

$$y^{(2)}_{ij} \geq C \sum_{p \in N(i) \cap N(j)} x_p + (x_i + x_j - 2), \qquad j > i, \ i, j \in V; \tag{17}$$

$$y^{(r)}_{ij} \leq x_i, \quad y^{(r)}_{ij} \leq \sum_{p \in N(i)} y^{(r-1)}_{pj}, r = 3, \ldots, s; \qquad j > i, \ i, j \in V; \tag{18}$$

$$y^{(r)}_{ij} \geq C \sum_{p \in N(i)} y^{(r-1)}_{pj} + (x_i - 1), \qquad j > i, \ i, j \in V; \tag{19}$$

$$x_i \in \{0, 1\}, \quad y^{(r)}_{ij} \in \{0, 1\}, \quad i, j \in V, r \in \{2, \ldots, s\}. \tag{20}$$

In this formulation, x is the characteristic vector of the dominating s-club. The binary variable $y^{(r)}_{ij}$ takes a value of 1 iff there is a path of length r from vertex i to vertex j in the dominating s-club.

C is a constant, which can be set to, e.g., $C = \frac{1}{n}$. As suggested by [34], setting $C = |N(i) \cap N(j)|^{-1}$ results in a tighter formulation. However, this is not applicable when $|N(i) \cap N(j)| = 0$. The objective is to minimize the cardinality of the dominating s-club. Constraint (14) ensures that each vertex is dominated. Proposition 2 and Remark 4 below will show that this classical domination constraint can be replaced with a stronger open neighborhood inequality $\sum_{j \in N(i)} x_j \geq 1$ (assuming that no single vertex dominates V). Constraint (15) ensures that the distance between dominating vertices $i, j \in V$ is at most s. Constraints (16) and (17) are for paths of length 2, whereas constraints (18) and (19) are for paths of length more than 2. We refer the reader to [34] for more detail on the s-club (diameter) constraints.

Next, we explore some valid inequalities for dominating s-club. Note that any feasible solution to dominating s-club is feasible to CDS. Accordingly, the convex hull of feasible solutions to dominating s-club is a subset of that of CDS. Therefore, an inequality which is valid for CDS is valid for dominating s-club. A subset of vertices in a connected graph is called a *cutset* if its removal disconnects the graph.

Proposition 2 *The following statements are valid for any characteristic vector x of a dominating s-club or a CDS.*

- **Cutset inequalities**. *For any cutset C, we have*

$$\sum_{i \in C} x_i \geq 1. \tag{21}$$

- **Open neighborhood inequalities.** *Suppose that no single vertex dominates V, then*

$$\sum_{j \in N(i)} x_j \geq 1 \quad i \in V. \tag{22}$$

- **Cut vertex fixing**. *If $i \in V$ is a cut vertex, then $x_i = 1$.*

Proof. The cutset inequalities have been established by [35]. Open neighborhood inequalities follow from the cutset inequalities because $N(i)$ is a cutset. Cut vertex fixing is given by [26]. It also follows from the cutset inequalities by letting $C = \{i\}$. □

It should be noted that [35], in fact, shows that the collection of all *minimal* cutset inequalities (along with with binary constraints on the variables) gives a proper formulation for CDS.

Remark 4 *The open neighborhood inequalities for dominating s-club and CDS subsume the classical domination constraints.*

The previous proposition shows how to create valid inequalities and fix certain variables for CDS (and consequently, dominating s-club). We can establish stronger, analogous results for dominating s-club.

Proposition 3 *Suppose a graph G has a dominating s-club, and let x be the characteristic vector of an arbitrary dominating s-club. Then the following statements hold.*

- **Diameter-critical set inequalities.** *If there is a subset of vertices $S \subseteq V$ such that $diam(G[V \setminus S]) \geq s + 3$, then*

$$\sum_{i \in S} x_i \geq 1. \tag{23}$$

- **Diameter-critical vertex fixing.** *If there is a vertex $i \in V$ such that $diam(G[V \setminus \{i\}]) \geq s+3$, then $x_i = 1$.*

Proof. We prove the diameter-critical set inequalities by contradiction. Suppose G has a dominating s-club $D' \subseteq V \setminus S$. Since D' dominates G, D' must also dominate $G[V \setminus S]$, which means that D' is a dominating s-club for $G[V \setminus S]$. By Proposition 1, $G[V \setminus S]$ has no dominating s-club for $s < diam(G[V \setminus S]) - 2$. However, $s \leq diam(G[V \setminus S]) - 3$, a contradiction. To establish diameter-critical vertex fixing, let $S = \{i\}$ and apply the diameter-critical set inequalities. □

Remark 5 *The diameter-critical set inequalities subsume the cutset inequalities.*

Proposition 4 (Variable fixing based on neighborhood inclusion) *The following statement is valid for both dominating s-club and CDS. If there exist $J \subset V$ and $I \subseteq V \setminus J$ such that $\forall i \in I \; \exists j \in J$ satisfying $N[i] \subseteq N[j]$ or $N(i) \subseteq N(j)$, then there exists an optimal solution with $x_i = 0 \; \forall i \in I$.*

Proof. Similar statements for closed neighborhood inclusion are well-known for the minimum dominating set problem [1] and the MCDS problem [26]. To prove the statement for the minimum dominating s-club problem, consider I and J satisfying the closed neighborhood inclusion. Suppose that for some $i \in I$, $x_i = 1$ in some feasible solution. Then for an arbitrary $j \in J$ such that $N[i] \subseteq N[j]$, x can be modified to obtain a feasible solution with the same objective value by setting $x_i = 0$ and $x_j = 1$ (if $x_j \neq 1$). All vertices that were dominated by i are now dominated by j. The diameter has not increased, as any shortest path through vertex i can use vertex j instead. This procedure can be repeated to obtain a feasible solution with $x_i = 0 \; \forall i \in I$.

The proof for the open neighborhood inclusion is the same, except that i will not be dominated by j when $x_j = 0$. However, by the open-neighborhood inequalities, there exists a neighbor i' of i with $x_{i'} = 1$. □

5 On the MCDS problem

Consider a generalization of the minimum dominating s-club problem in which we let the parameter s depend on some input, such as the number n of vertices in the graph G. Then the MCDS problem can be solved via the minimum dominating s-club problem. In the naive approach, simply let $s = n - 1$. However, more practical ways of choosing s may exist.

Proposition 5 *Suppose U is an upper bound on $\gamma_c(G)$. A solution to the minimum dominating s-club problem, where $s = (U - 1)$, solves the MCDS problem.*

Proof. An optimal CDS has at most U nodes, thus its diameter is at most $U - 1$. □

Proposition 6 *Suppose P solves the minimum dominating s-club problem in G. If $|P| \leq s + 2$, then P solves the MCDS problem in G.*

Proof. Suppose P is an optimal solution of the minimum dominating s-club problem that is not optimal for the MCDS problem. Then there exists a CDS P^* with $|P^*| \leq |P| - 1$. Then $diam(G[P^*]) \leq |P^*| - 1 \leq |P| - 2 \leq s$. Therefore P^* is a dominating s-club of size less than $|P|$, contradicting the optimality of P. □

The last two propositions lead to two different ways to solve the MCDS problem. In the first approach (using Proposition 5), use any MCDS heuristic to find a decent upper bound U on $\gamma_c(G)$. Then solve the minimum dominating s-club problem with $s = U - 1$ to solve the MCDS problem.

In the second approach, use Proposition 6 and solve a series of minimum dominating s-club problems for various values of s. Small values of s, which typically warrant applicability of effective scale-reduction techniques, may not give a small enough dominating s-club to apply Proposition 6. In a simple approach, start with $s = diam(G) - 2$ and solve the minimum dominating s-club problem. If Proposition 6 applies, then we have an optimal solution to the MCDS problem. Otherwise, increment s by 1 and repeat. This approach has the advantage that an optimal solution for one stage constitutes a feasible starting solution for the subsequent problem. On the other hand, Proposition 6 may not be applicable for any value of $s < \gamma_c(G) - 1$, potentially resulting in an excessive number of stages if $(\gamma_c(G) - diam(G))$ is large.

Our preliminary numerical experiments with these approaches yielded results that were generally outperformed by other methods for MCDS proposed in the literature [26, 13], especially on instances with large diameter. However, our results were competitive on instances with a small CDS. We leave an in-depth investigation of the suitability of the proposed formulation for the MCDS problem for future work.

6 Experimental results

All computational experiments were conducted on *Dell Precision Workstation T7500®* computers, each with eight 2.40 GHz Intel Xeon® processors, and 12 GB RAM. The solver used was ILOG CPLEX 12.1®. Before sending an instance to the solver, we checked if a single vertex or pair of adjacent vertices dominated, in which case the problem is solved. If neither of these conditions were satisfied, then a heuristic CDS solution was generated as described by [10]. This CDS was used as the initial feasible solution for the minimum dominating s-club problem if it satisfied the diameter constraint.

In the computational experiments, variable fixing based on neighborhood inclusion was applied in the following way. Let $V = \{1, \ldots, n\}$. Then if the closed (open) neighborhood of i is a strict subset of the closed (open) neighborhood of j, we set $x_i = 0$. If the closed (open) neighborhoods of i and j are equal, then fix $x_{\min\{i,j\}} = 0$. Then by Proposition 4 there exists an optimal solution that remains feasible after this variable-fixing procedure.

A motivating application for dominating s-club occurs in wireless sensor networks, where nodes can communicate with each other when they are within the transmission range (unit distance) of each other. This leads to a graph-theoretic model called a unit disk graph, where nodes lie in the Euclidean plane and an edge joins two nodes if the Euclidean distance between them is below the unit distance (i.e., one node is in the unit disk centered at the other node). We created several random unit disk graphs, where the points lie in a 2-dimensional box, and the number of points and communication threshold are specified by a set of parameters. Each point was generated from a random 'x' and 'y' coordinate, each uniformly distributed between 0 and the length of the box side. If the unit disk graph generated by these points was not connected, it was discarded and another graph was created. The parameters used to generate these graphs are the same used by [10].

The aim of our first set of experiments was to explore the value of employing the variable fixing rules and valid inequalities described in Sec. 4. For this purpose, we used a set of 100-vertex random unit disk graphs with the length of the box side set at 100 and the unit disk radius ranging between 20 and 50, with the increment of 5. In addition to executing the basic version of formulation (14)–(20) (labeled "Basic" in the table), we considered the following variations of the IP approach.

- Add diameter-critical vertex fixing to the basic implementation ("DCF");

Table 2: Summary of the results for random UDGs with $s = diam(G) - 2$. The first column, 'r', contains the unit radius used to generate a UDG. Twenty random UDGs have been considered for each radius value. There are two columns, labeled '#' and 'time' for each of the five methods considered, the first column containing the number of instances (out of 20) for which the method terminated within the 3600-second time limit, and the second column reporting the average running time, in seconds, taken by the method for each set of UDG instances. (If an instance was not solved within the time limit, a time of 3600 seconds is used when computing the average.) If the time limit was reached for at least one instance from the set, the corresponding average time is shown in parentheses.

r	Basic		DCF		NIF		ONI		All	
	#	ave time	#	ave time	#	ave time	#	ave time	#	ave time
20	1	(3423.9)	1	(3422.2)	7	(2867.1)	1	(3424.6)	11	(2652.7)
25	2	(3371.2)	2	(3345.7)	10	(2246.1)	2	(3365.0)	13	(2194.7)
30	13	(1604.6)	13	(1574.4)	20	476.2	13	(1411.9)	20	284.7
35	17	(1465.2)	17	(1462.5)	20	34.6	18	(1154.1)	20	34.2
40	20	5.9	20	6.7	20	2.0	20	6.4	20	2.9
45	20	4.9	20	6.0	20	1.8	20	6.1	20	2.7
50	20	1.7	20	2.9	20	1.5	20	1.7	20	2.7

- Add variable fixing based on neighborhood inclusion to the basic implementation ("NIF");
- Replace the standard domination constraints (15) with the open neighborhood inequalities (22) ("ONI");
- Replace the standard domination constraints (15) with the open neighborhood inequalities (22), and add diameter-critical vertex fixing, as well as variable fixing based on neighborhood inclusion to the basic implementation ("All").

A summary of the results comparing the different variations of the IP approach on 100-vertex random UDGs with $s = diam(G) - 2$ is presented in Table 2, with detailed results provided in Table 6 of the appendix. As can be seen from the tables, it is typically most beneficial to include all three of the considered enhancements to the basic implementation, therefore this was made the case in our remaining experiments reported in Tables 3–5. The setting "NIF" also works pretty well, which we suspect is because this can reduce the dimension of the problem (particularly, when fixing $x_i = 0$ for a vertex i that lies near an edge of the box).

We tested the approach using three of the lowest feasible values of s for each graph, namely $s = diam(G) - 2, diam(G) - 1, diam(G)$. When the problem was not solved to optimality in 3600 seconds, then the size of the best found dominating s-club is reported in all three tables. In addition, the optimality gap, as reported by CPLEX, is given in the column labeled "gap" for each run.

Tables 3 and 4 present the results of experiments for 100-vertex and 150-vertex instances of random unit disk graphs, respectively. The first four columns of these tables contain the dimensions of the box used to generate the graphs, the radius or 'communication' distance, the density, and the diameter of the graph, respectively. The remaining nine columns contain the size of the best found dominating s-club, the CPU time (in seconds), and the optimality gap (as reported by CPLEX), respectively, for each of the considered values of s.

Table 5 reports the results of experiments on a set of MCDS instances from [26] and [32]. In this table, the symbol † indicates that the solver (with default settings) did not solve the LP relaxation

Table 3: Results for random unit disk graphs with $n = 100$.

				$s = diam(G)$			$s = diam(G) - 1$			$s = diam(G) - 2$		
dimensions	radius	$\rho(G)$	$diam(G)$	$\gamma^s_{club}(G)$	time	gap	$\gamma^s_{club}(G)$	time	gap	$\gamma^s_{club}(G)$	time	gap
100×100	20	0.10	9	≤ 20	> 3600	0.34	≤ 21	> 3600	0.37	19	> 3600	0.13
	25	0.16	6	≤ 11	> 3600	0.27	≤ 11	> 3600	0.24	12	206.4	0
	30	0.24	5	8	2538.0	0	8	406.0	0	9	2.8	0
	35	0.26	4	6	102.0	0	6	4.4	0	6	1.9	0
	40	0.33	4	5	4.2	0	5	4.1	0	5	2.5	0
	45	0.46	4	4	5.6	0	4	3.8	0	4	2.3	0
	50	0.48	3	3	3.8	0	3	2.2	0	3	2.7	0
120×120	20	0.08	12	≤ 25	> 3600	0.29	≤ 26	> 3600	0.26	≤ 24	> 3600	0.09
	25	0.10	9	≤ 21	> 3600	0.37	≤ 22	> 3600	0.36	≤ 20	> 3600	0.09
	30	0.14	7	≤ 12	> 3600	0.25	≤ 12	> 3600	0.25	12	230.2	0
	35	0.19	5	≤ 10	> 3600	0.20	9	732.5	0	11	2.4	0
	40	0.24	5	≤ 7	> 3600	0.14	7	979.5	0	7	13.0	0
	45	0.35	4	5	107.9	0	5	3.6	0	5	2.6	0
	50	0.39	4	4	4.3	0	4	3.7	0	4	2.7	0
140×140	30	0.13	7	≤ 15	> 3600	0.27	≤ 15	> 3600	0.24	15	715.5	0
	35	0.16	6	≤ 13	> 3600	0.40	≤ 11	> 3600	0.24	12	79.7	0
	40	0.19	6	≤ 9	> 3600	0.22	≤ 10	> 3600	0.30	9	4.1	0
	45	0.26	5	7	3452.8	0	7	1097.0	0	7	15.7	0
	50	0.32	4	5	418.5	0	5	4.7	0	5	2.3	0
	55	0.37	4	5	97.8	0	5	3.8	0	5	2.9	0
	60	0.40	3	4	3.1	0	4	2.8	0	4	2.4	0
160×160	30	0.09	11	≤ 22	> 3600	0.36	≤ 22	> 3600	0.34	≤ 21	> 3600	0.22
	35	0.11	8	≤ 20	> 3600	0.44	≤ 60	> 3600	0.81	≤ 17	> 3600	0.16
	40	0.16	6	≤ 12	> 3600	0.29	≤ 11	> 3600	0.17	12	245.6	0
	45	0.21	6	≤ 12	> 3600	0.42	≤ 66	> 3600	0.89	10	2867.5	0
	50	0.22	5	6	184.6	0	6	152.6	0	6	3.1	0
	55	0.27	5	6	196.7	0	6	1157.4	0	6	18.0	0
	60	0.32	4	5	4.4	0	5	4.1	0	5	2.7	0

within the time limit, and an interior point method was used to find an appropriate gap. An asterisk $*$ indicates that no solution was found in the time limit, and the gap could not be calculated. When the instance was shown to be infeasible in the time limit, the size of the dominating s-club is listed as ∞.

As one can observe from the tables, the best results in terms of solution time occur when s is small and when the edge density of a graph is high. Each of the 100-vertex instances was solved to optimality when the edge density was at least 25%. Similarly, each of the 150 vertex instances was solved to optimality for densities of at least 30%. However, sparse instances appear to be more difficult to solve. This has also been noted for the MCDS problem [32] and for the maximum s-club problem [34].

Table 4: Results for random unit disk graphs with $n = 150$.

				$s = diam(G)$			$s = diam(G) - 1$			$s = diam(G) - 2$		
dimensions	radius	$\rho(G)$	$diam(G)$	$\gamma^s_{club}(G)$	time	gap	$\gamma^s_{club}(G)$	time	gap	$\gamma^s_{club}(G)$	time	gap
120×120	50	0.39	4	4	19.1	0	4	15.1	0	4	11.4	0
	55	0.44	3	4	14.2	0	4	8.0	0	4	10.7	0
	60	0.51	3	3	17.1	0	3	14.9	0	3	12.3	0
	65	0.52	3	3	17.0	0	3	9.4	0	3	12.4	0
	70	0.61	3	2	0.0	0	2	0.0	0	2	0.0	0
	75	0.63	3	2	0.0	0	2	0.0	0	2	0.0	0
	80	0.72	2	2	0.0	0	2	0.0	0			
140×140	50	0.30	4	≤ 7	> 3600	0.29	5	16.6	0	6	9.0	0
	55	0.34	4	4	17.4	0	4	11.2	0	5	10.9	0
	60	0.35	4	4	15.3	0	4	14.0	0	4	10.1	0
	65	0.43	4	4	19.1	0	4	13.4	0	4	12.5	0
	70	0.47	3	3	14.9	0	3	8.5	0	4	11.3	0
	75	0.53	3	2	0.0	0	2	0.0	0	2	0.0	0
	80	0.57	3	2	0.0	0	2	0.0	0	2	0.0	0
160×160	50	0.24	5	≤ 9	> 3600	0.33	≤ 8	> 3600	0.25	7	1552.9	0
	55	0.29	4	≤ 6	> 3600	0.17	6	241.6	0	6	8.4	0
	60	0.29	4	5	196.6	0	5	135.6	0	5	8.9	0
	65	0.35	4	4	138.9	0	4	16.5	0	5	12.6	0
	70	0.43	4	4	20.8	0	4	18.5	0	4	12.7	0
	75	0.42	3	3	12.7	0	3	7.3	0	4	10.4	0
	80	0.51	3	3	17.1	0	3	9.4	0	3	12.2	0
180×180	50	0.18	6	≤ 13	> 3600	0.46	≤ 13	> 3600	0.46	≤ 11	> 3600	0.36
	55	0.21	5	≤ 9	> 3600	0.33	≤ 10	> 3600	0.4	8	8.6	0
	60	0.24	5	≤ 9	> 3600	0.39	≤ 8	> 3600	0.33	7	296.1	0
	65	0.26	4	5	1382.5	0	5	69.6	0	6	8.5	0
	70	0.32	4	5	517.7	0	5	60.8	0	6	10.7	0
	75	0.34	4	4	13.4	0	4	9.9	0	5	10.3	0
	80	0.38	4	4	17.4	0	4	16.2	0	4	11.7	0

Table 5: Test results for instances from [26] and [32]. Blank entries indicate that $s = diam(G) - 2 < 1$.

			$s = diam(G)$			$s = diam(G) - 1$			$s = diam(G) - 2$		
n	$\rho(G)$	$diam(G)$	$\gamma^s_{club}(G)$	time	gap	$\gamma^s_{club}(G)$	time	gap	$\gamma^s_{club}(G)$	time	gap
30	0.10	8	15	1.5	0	15	0.7	0	∞	0.4	0
	0.20	5	7	6.2	0	7	0.6	0	8	0.7	0
	0.30	3	4	1.0	0	5	0.6	0	∞	0.3	0
	0.50	2	3	0.6	0	3	0.3	0			
	0.70	2	2	0.0	0	2	0.0	0			
50	0.05	14	≤ 31	> 3600	0.05	32	1598.0	0	32	182.7	0
	0.10	5	13	487.1	0	14	33.2	0	19	1.3	0
	0.20	3	7	30.8	0	7	1.2	0	∞	0.3	0
	0.30	3	5	1.8	0	5	2.2	0	∞	0.4	0
	0.50	2	3	1.2	0	3	0.4	0			
	0.70	2	2	0.0	0	2	0.0	0			
70	0.05	8	≤ 32	> 3600	0.25	≤ 29	> 3600	0.10	32	249.8	0
	0.10	4	13	1050.2	0	17	55.5	0	∞	3.4	0
	0.20	3	7	162.1	0	8	8.6	0	∞	0.6	0
	0.30	3	5	436.4	0	5	6.1	0	∞	0.8	0
	0.50	2	3	2.6	0	3	1.0	0			
	0.70	2	2	0.0	0	2	0.0	0			
100	0.05	5	≤ 34	> 3600	0.39	40	3316.2	0	∞	8.8	0
	0.10	4	≤ 15	> 3600	0.24	≤ 18	> 3600	0.34	∞	15.7	0
	0.20	3	≤ 10	> 3600	0.42	9	305.6	0	∞	1.5	0
	0.30	2	6	48.2	0	∞	2.9	0			
	0.50	2	4	16.6	0	4	3.2	0			
	0.70	2	3	26.1	0	3	4.1	0			
120	0.05	6	≤ 30	> 3600	$0.25^\dagger$	≤ 34	> 3600	0.34	≤ 41	> 3600	0.37
	0.10	3	≤ 20	> 3600	0.43	∞	43.8	0	∞	24.2	0
	0.20	3	≤ 10	> 3600	0.41	≤ 10	> 3600	0.23	∞	2.7	0
	0.30	3	≤ 8	> 3600	0.50	6	280.0	0	∞	4.8	0
	0.50	2	4	49.2	0	4	6.1	0			
	0.70	2	3	645.2	0	3	7.7	0			
150	0.05	5	≤ 37	> 3600	$0.42^\dagger$	≤ 39	> 3600	0.44	∞	66.0	0
	0.10	3	≤ 22	> 3600	0.47	∞	260.5	0	∞	97.2	0
	0.20	3	≤ 13	> 3600	0.56	≤ 12	> 3600	0.45	∞	6.0	0
	0.30	2	6	841.9	0	∞	12.9	0			
	0.50	2	4	144.2	0	4	13.2	0			
	0.70	2	3	2187.2	0	3	17.9	0			
200	0.05	4	≤ 46	> 3600	$0.51^\dagger$	$*$	> 3600	$*$	∞	224.6	0
	0.10	3	≤ 25	> 3600	0.58	$*$	> 3600	$*$	∞	399.0	0
	0.20	3	≤ 19	> 3600	0.73	≤ 13	> 3600	0.57	∞	18.5	0
	0.30	2	≤ 7	> 3600	0.48	∞	72.2	0			
	0.50	2	4	1635.9	0	4	41.5	0			
	0.70	2	≤ 4	> 3600	0.50	3	53.8	0			

7 Conclusion

We consider the problem of finding a smallest diameter-restricted dominating set (i.e., dominating s-club). We show that determining if a diameter $s+1$ graph admits a dominating s-club is NP-complete for any fixed $s \geq 1$, filling a gap in the knowledge about the problem's complexity. Moreover, we remark that (for $s \geq 2$) the minimum dominating s-club problem remains hard to approximate within a logarithmic factor even when a dominating s-club is known to exist. We provide a compact integer programming formulation for the minimium dominating s-club problem and develop some variable fixing rules and valid inequalities.

In the future, one may seek to make the dominating s-club robust under edge and vertex failures. [34] explore various ways to ensure robustness of the s-club. One proposed structure, an R-robust s-club, not only remains connected after $R-1$ vertex failures, but also maintains diameter s. Similar avenues may be pursued to make the dominating s-club more robust. An interesting concept to explore may be an R-robust dominating s-club: a structure which maintains its dominating nature and diameter s despite the failure of $R-1$ dominating vertices.

The complexity of the minimum dominating s-club problem in unit disk graphs is an open question. Since the MCDS problem remains hard in this class of graphs, we suspect that it also remains challenging. Other authors (e.g., [22]) have proposed approximation algorithms for similar problems in disk graphs, providing evidence that the problem remains difficult, or that an appropriate NP-completeness reduction would be nontrivial.

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A NP-completeness reductions for Theorem 2

Lemma 1 *For any fixed positive* **odd** *integer s, the problem of determining if a graph of diameter $s+1$ has a dominating s-club is NP-complete.*

Proof. The reduction is from Dominating Clique in a diameter two graph $G = (V, E)$. The case $s = 1$ is exactly Dominating Clique from Theorem 1, so consider $s \geq 3$ odd. We construct a graph $G' = (V', E')$ of diameter $s+1$ that has a dominating s-club if and only if G has a dominating clique.

First define the vertex set $V' = V^1 \cup \cdots \cup V^{s+2}$ where $V^i = \{j^{(i)} : j \in V\},\ i = 1, \ldots, s+2$. Thus we have $s+2$ copies of the vertex set V. Now we define the edge set $E' = E_1 \cup E_2 \cup E_3$, where

$$E_1 = \bigcup_{i=1}^{s+1} \bigcup_{j \in V} \{j^{(i)}, j^{(i+1)}\} \tag{24}$$

$$E_2 = \bigcup_{i \in \{1, \frac{s+3}{2}, s+2\}} \{\{j^{(i)}, k^{(i)}\} : \{j, k\} \in E\} \tag{25}$$

$$E_3 = \bigcup_{i \in \{1, s+1\}} \{\{j^{(i)}, k^{(i+1)}\} : \{j, k\} \in E\}. \tag{26}$$

The reduction is polynomial since the number of vertices in G' is $|V'| = (s+2)|V|$ and s is a constant.

Claim 0: $diam(G') = s+1$. First see that $diam(G') \geq s+1$ since vertices in V^1 and V^{s+2} are distance $s+1$ apart. Now we consider four cases where we traverse from $j^{(i)} \leftrightarrow q^{(p)}$, showing that in each case there exists a path of length at most $s+1$.

1. $V^2 \cup \cdots \cup V^{s+1} \leftrightarrow V^2 \cup \cdots \cup V^{s+1}$ and $\{j, q\} \in E$. Consider the path $(j^{(i)}, \ldots, j^{(\frac{s+3}{2})}, q^{(\frac{s+3}{2})}, \ldots, q^{(p)})$ of length at most $(\frac{s+3}{2} - 2) + 1 + (\frac{s+3}{2} - 2) = s$.
2. $V^2 \cup \cdots \cup V^{s+1} \leftrightarrow V^2 \cup \cdots \cup V^{s+1}$ and $\{j, q\} \notin E$. Since $diam(G) = 2$, vertices j and q must share a common neighbor c in G. Consider the path $(j^{(i)}, \ldots, j^{(\frac{s+3}{2})}, c^{(\frac{s+3}{2})}, q^{(\frac{s+3}{2})}, \ldots, q^{(p)})$ of length at most $(\frac{s+3}{2} - 2) + 2 + (\frac{s+3}{2} - 2) = s+1$.
3. $V^1 \leftrightarrow V^1$ or $V^{s+2} \leftrightarrow V^{s+2}$. Note that $G'[V^1] \simeq G'[V^{s+2}] \simeq G$ so $diam(G'[V^1]) = diam(G'[V^{s+2}]) = 2$.
4. $V^1 \leftrightarrow V^{s+2}$. Without loss of generality, we can assume that $i = 1$ and $p = s+2$. If j and q are adjacent in G or if $j = q$, then consider the path $(j^{(1)}, q^{(2)}, \ldots, q^{(s+2)})$ of length $s+1$. If j and q are nonadjacent in G, then j and q share a common neighbor c in G and consider the path $(j^{(1)}, c^{(2)}, \ldots, c^{(s+1)}, q^{(s+2)})$ of length $s+1$.

Claim 1: if G has a dominating clique D, then G' has a dominating s-club. We claim that $D' = \{j^{(i)} : j \in D, i \in \{2, s+1\}\} \cup V^3 \cup \cdots \cup V^s$ is a dominating s-club for G'. First see that D' is dominating. Every vertex from $V^2 \cup \cdots \cup V^{s+1}$ is easily seen to be dominated by D'. Vertices in V^1 (resp., V^{s+2}) will be dominated by $D' \cap V^2$ (resp., $D' \cap V^{s+1}$) through the edge subsets E_1 and E_3. We show that $diam(G'[D']) \leq s$, by considering a pair $j^{(i)}, q^{(p)}$ of vertices from D', and showing that there exists a path between them of length at most s in $G'[D']$. We consider three cases.

1. If $q = j$, consider the path $(j^{(i)}, \ldots, j^{(p)})$ of length at most $s-1$.

2. If $\{j, q\} \in E$, consider the path $(j^{(i)}, \ldots, j^{(\frac{s+3}{2})}, q^{(\frac{s+3}{2})}, \ldots, q^{(p)})$. The length is at most $(\frac{s+3}{2} - 2) + 1 + (\frac{s+3}{2} - 2) = s$.

3. Finally, if $\{j, q\} \notin E$, consider the path $(j^{(i)}, \ldots, j^{(\frac{s+3}{2})}, c^{(\frac{s+3}{2})}, q^{(\frac{s+3}{2})}, \ldots, q^{(p)})$, where c is a common neighbor of j and q in G. Now, since $\{j, q\} \notin E$, either $3 \leq i \leq s$ or $3 \leq p \leq s$ (since if both vertices belong to $D' \cap (V^2 \cup V^{s+1})$ then they would be adjacent in G by construction of D'). This ensures that the path's length is either at most $(\frac{s+3}{2} - 3) + 2 + (\frac{s+3}{2} - 2) = s$ or at most $(\frac{s+3}{2} - 2) + 2 + (\frac{s+3}{2} - 3) = s$.

Claim 2: if G' has a dominating s-club D', then G has a dominating clique. See that we cannot simultaneously have $D' \cap V^1 \neq \emptyset$ and $D' \cap V^{s+2} \neq \emptyset$, since vertices from these sets are distance $s+1$ apart. Without loss of generality, we can assume that $D' \cap V^{s+2} = \emptyset$. We claim that $D = \{j : j^{(s+1)} \in D' \cap V^{s+1}\}$ is a dominating clique for G. First see that D dominates G; every vertex j^{s+2} in G' was dominated by a vertex $k^{(s+1)}$ in G', so k will dominate j in G. Now we show that D is a clique in G. Consider a pair $j^{(s+1)}, k^{(s+1)}$ of distinct vertices from $D' \cap V^{s+1}$. The only possible path of length at most s between them that does not use vertices from V^{s+2} is $(j^{(s+1)}, j^{(s)}, \ldots, j^{(\frac{s+3}{2})}, k^{(\frac{s+3}{2})}, \ldots, k^{(s)}, k^{(s+1)})$, meaning that $j^{(\frac{s+2}{2})}, k^{(\frac{s+2}{2})}$ must be adjacent in G'; thus j and k are adjacent in G by the edge subset E_2. □

Lemma 2 *For any fixed positive* ***even*** *integer s, the problem of determining if a graph of diameter $s+1$ has a dominating s-club is NP-complete.*

Proof. The reduction is from Dominating Clique in a diameter two graph $G = (V, E)$. Let $s \geq 2$ be even. We construct a graph $G' = (V', E')$ of diameter $s+1$ that has a dominating s-club if and only if G has a dominating clique.

First define the vertex set $V' = E \cup V^1 \cup \cdots \cup V^{s+2}$ where $V^i = \{j^{(i)} : j \in V\},\ i = 1, \ldots, s+2$. Thus we have $s+2$ copies of the vertex set V, and the edge set E from G serves as a vertex subset of G'. Now we define the edge set $E' = E_1 \cup \cdots \cup E_5$, where

$$E_1 = \bigcup_{i=1}^{s+1} \bigcup_{j \in V} \left\{ j^{(i)}, j^{(i+1)} \right\} \tag{27}$$

$$E_2 = \bigcup_{i \in \{1, s+2\}} \left\{ \{j^{(i)}, k^{(i)}\} : \{j, k\} \in E \right\} \tag{28}$$

$$E_3 = \bigcup_{i \in \{1, s+1\}} \left\{ \{j^{(i)}, k^{(i+1)}\} : \{j, k\} \in E \right\} \tag{29}$$

$$E_4 = \left\{ \{e, e'\} : e, e' \in E, e \neq e' \right\} \tag{30}$$

$$E_5 = \left\{ \{j^{(p)}, e\} : e = \{j, k\} \in E, p \in \left\{ \frac{s}{2} + 1, \frac{s}{2} + 2 \right\} \right\}. \tag{31}$$

The reduction is polynomial since the number of vertices in G' is $|V'| = (s+2)|V| + |E|$ and s is a constant. Define a two-valued function $f(k) = \frac{s}{2} + 1$ whenever $k \leq \frac{s}{2} + 1$ and $f(k) = \frac{s}{2} + 2$ otherwise. An important property of $f(\cdot)$ is that $|f(k) - k| \leq \frac{s}{2} - 1$ for any $2 \leq k \leq s+1$.

Claim 0: $diam(G') = s+1$. First see that $diam(G') \geq s+1$ since vertices in V^1 and V^{s+2} are distance $s+1$ apart. We consider four cases, showing that in each case there exists a path of length at most $s+1$.

1. $E \leftrightarrow V^1 \cup \cdots \cup V^{s+2}$. Let the endpoints of the path be $e = \{j, k\}$ and $q^{(p)}$.

(a) If $q = j$ or $q = k$, consider the path $(e, q^{(f(p))}, \ldots, q^{(p)})$ of length at most $\frac{s}{2} + 1$.

(b) In the other case, there will be an edge $e' = \{q, t\} \in E$. Then consider the path $(e, e', q^{(f(p))}, \ldots, q^{(p)})$ of length at most $\frac{s}{2} + 2$ (which is at most $s + 1$ for any $s \geq 2$).

2. $V^2 \cup \cdots \cup V^{s+1} \leftrightarrow V^2 \cup \cdots \cup V^{s+1}$. Let the endpoints of the path be $j^{(i)}$ and $q^{(p)}$.

   (a) If $\{j, q\} \in E$, consider the path $(j^{(i)}, \ldots, j^{(f(i))}, \{j, q\}, q^{(f(p))}, \ldots, q^{(p)})$ of length at most $(\frac{s}{2} - 1) + 2 + (\frac{s}{2} - 1) = s$.

   (b) If $\{j, q\} \notin E$, then j and q share a common neighbor c in G. Then consider the path $(j^{(i)}, \ldots, j^{(f(i))}, \{j, c\}, \{c, q\}, q^{(f(p))}, \ldots, q^{(p)})$ of length at most $(\frac{s}{2}-1)+3+(\frac{s}{2}-1) = s+1$.

3. $V^1 \leftrightarrow V^1$ or $V^{s+2} \leftrightarrow V^{s+2}$. Note that $G'[V^1] \simeq G'[V^{s+2}] \simeq G$ so $diam(G'[V^1]) = diam(G'[V^{s+2}]) = 2$.

4. $V^1 \leftrightarrow V^{s+2}$. Without loss of generality, we can assume that $i = 1$ and $p = s + 2$. If j and q are adjacent in G or if $j = q$, then consider the path $(j^{(1)}, q^{(2)}, \ldots, q^{(s+2)})$ of length $s + 1$. If j and q are nonadjacent in G, then they share a common neighbor c, and consider the path $(j^{(1)}, c^{(2)}, \ldots, c^{(s+1)}, q^{(s+2)})$ of length $s + 1$.

Claim 1: if G has a dominating clique D, then G' has a dominating s-club. We claim that $D' = E \cup Q \cup \{j^{(i)} : j \in D, i \in \{2, s+1\}\}$ is a dominating s-club for G', where $Q = \emptyset$ for $s = 2$ and $Q = V^3 \cup \cdots \cup V^s$ otherwise. First see that D' is dominating. Every vertex from $E \cup V^2 \cup \cdots \cup V^{s+1}$ is easily seen to be dominated by D'. Vertices in V^1 (resp., V^{s+2}) will be dominated by $D' \cap V^2$ (resp., $D' \cap V^{s+1}$) through the edge subsets E_1 and E_3. We show that $diam(G'[D']) \leq s$, by considering a pair $j^{(i)}, q^{(p)}$ of vertices from D', and showing that there exists a path between them of length at most s in $G'[D']$. It is easy to see that if either of the vertices belongs to E, then we are done because the path of length at most s from Claim 0 remains intact. In the other case, let the two vertices be $j^{(i)}, q^{(p)} \in D' \setminus E$.

1. If $j^{(i)}, q^{(p)} \in Q$, then we are done since $Q \cup E \subset D'$ and $diam(G'[Q \cup E]) = s - 1$.

2. If one of them belongs to Q, then traverse from this vertex to the neighbor in Q. All other vertices from Q can be visited from this neighbor in at most $s - 1$ hops, so the total distance is at most s.

3. Otherwise, $j^{(i)}, q^{(p)} \in V^2 \cup V^{s+1}$ and $\{j, q\} \in E$. Consider the path $(j^{(i)}, \ldots, j^{(f(i))}, \{j, q\}, q^{(f(p))}, \ldots, q^{(p)})$ of length at most $(\frac{s}{2} - 1) + 2 + (\frac{s}{2} - 1) = s$.

Claim 2: if G' has a dominating s-club D', then G has a dominating clique. See that we cannot simultaneously have $D' \cap V^1 \neq \emptyset$ and $D' \cap V^{s+2} \neq \emptyset$, since vertices from these sets are distance $s + 1$ apart. Without loss of generality, we can assume that $D' \cap V^{s+2} = \emptyset$. We claim that $D = \{j : j^{(s+1)} \in D' \cap V^{s+1}\}$ is a dominating clique for G. First see that D dominates G; every vertex $j^{(s+2)}$ in G' was dominated by a vertex $k^{(s+1)}$ in G', so k will dominate j in G. Now we show that D is a clique in G. Consider a pair $j^{(s+1)}, k^{(s+1)}$ of distinct vertices from D'. The only possible path of length at most s between them that does not use vertices from V^{s+2} is $(j^{(s+1)}, j^{(s)}, \ldots, j^{(\frac{s}{2}+2)}, \{j, k\}, k^{(\frac{s}{2}+2)}, \ldots, k^{(s)}, k^{(s+1)})$, meaning that the edges $\{j^{(\frac{s}{2}+2)}, \{j, k\}\}$ and $\{\{j, k\}, k^{(\frac{s}{2}+2)}\}$ must belong to E', thus j and k are adjacent in G. ☐

B Cost of a more restrictive diameter for Remark 2

The following construction demonstrates the claim from Remark 2. Given positive integers q and s, we construct a graph $G = (V, E)$ for which $\gamma^s_{club}(G) \geq q$ and $\gamma^{s+1}_{club}(G) \leq s + 2$. The idea is to create large sets Y^0, Y^1 of vertices that must be dominated. Whenever we require diameter $s + 1$, the small set X will dominate them. However, when the diameter is restricted to s, X does not satisfy the diameter constraint, and a large number of vertices from $Z^0 \cup Z^s$ will be needed.

Define the vertex set $V = X \cup Y^0 \cup Y^1 \cup Z^0 \cup \cdots \cup Z^s$, where

$$X = \{x_i : 0 \leq i \leq s + 1\} \tag{32}$$

$$Y^i = \{y_{ij} : 1 \leq j \leq q\}, \text{ for } i = 0, 1 \tag{33}$$

$$Z^i = \begin{cases} \{z_{ij} : 1 \leq j \leq q\} & \text{if } i \in \{0, s\} \\ \{z_{i1}\} & \text{if } 1 \leq i \leq s - 1. \end{cases} \tag{34}$$

Now we construct the edge set $E = E_1 \cup \cdots \cup E_7$, where

$$E_1 = \{\{x_i, x_{i+1}\} : 0 \leq i \leq s\} \tag{35}$$

$$E_2 = \{x_0\} \times (Y^0 \cup Z^0) \tag{36}$$

$$E_3 = \{x_{s+1}\} \times (Y^1 \cup Z^s) \tag{37}$$

$$E_4 = \bigcup_{i=1}^{s} \{x_i\} \times Z^i \tag{38}$$

$$E_5 = \bigcup_{i=0}^{s-1} Z^i \times Z^{i+1} \tag{39}$$

$$E_6 = \{\{z_{0j}, z_{0k}\} : 1 \leq j < k \leq q\} \cup \{\{z_{sj}, z_{sk}\} : 1 \leq j < k \leq q\} \tag{40}$$

$$E_7 = \{\{y_{0j}, z_{0j}\} : 1 \leq j \leq q\} \cup \{\{y_{1j}, z_{sj}\} : 1 \leq j \leq q\}. \tag{41}$$

Thus, $G[X] \simeq P_{s+2}$, $diam(P_{s+2}) = s + 1$, and X is dominating, so $\gamma^{s+1}_{club}(G) \leq s + 2$. Also, see that $Z^0 \cup \cdots \cup Z^s$ is a dominating s-club, so $\gamma^s_{club}(G)$ is well-defined.

Notice that vertices x_0 and x_{s+1} cannot belong to the same s-club, since they are distance $s + 1$ apart. If x_0 does not belong to the dominating s-club S, then all vertices from Z^0 must belong to S in order to dominate Y^0. Similarly, if x_{s+1} does not belong to S, then all vertices from Z^s must belong to S to dominate Y^1. Thus $\gamma^s_{club}(G) \geq q$.

C Extended computational results

Table 6: Detailed results of experiments for random unit disk graphs (UDG). All considered graphs have 100 vertices that are uniformly distributed at random in a 100×100 square on the plane. The number after 'r' in each graph's name in the first column is the unit radius used to generate the corresponding UDG. The columns labeled 'm' and 'ρ' contain the number of edges and edge density, respectively. The next two columns contain the value of s, which is equal to $diam(G) - 2$ for each graph G, and the size γ^s_{club} of a minimum dominating s-club. In cases where γ^s_{club} is unknown, the best found upper bound is listed in parentheses. If a dominating s-club does not exist, $\gamma^s_{club} = \infty$ is reported. An asterisk * indicates that no solution was found within the time limit. Finally, the last five columns contain the running time (in sec.) for the five considered methods whenever the corresponding method terminated within 3600-second time limit, or the optimality gap reported by CPLEX after 3600 seconds (in parentheses), otherwise. Infinite gap indicates that no feasible solution was found by the corresponding method.

					Time (or gap)				
Graph	m	ρ	s	γ^s_{club}	Basic	DCF	NIF	ONI	All
r20_0	499	0.101	7	(18)	*	*	(0.330)	*	(0.181)
r20_1	493	0.100	6	21	(0.328)	(0.250)	2296.470	(0.305)	439.863
r20_2	503	0.102	7	18	*	*	2649.400	*	2295.460
r20_3	559	0.113	7	20	*	*	(0.153)	(0.526)	2709.580
r20_4	490	0.099	8	(23)	*	*	(0.388)	*	(0.361)
r20_5	475	0.096	7	(21)	*	(0.607)	(0.394)	*	(0.307)
r20_6	524	0.106	6	(20)	*	*	(0.161)	*	(0.150)
r20_7	538	0.109	6	18	*	*	3558.420	*	3110.290
r20_8	465	0.094	7	(24)	*	*	(0.199)	*	(0.244)
r20_9	519	0.105	6	∞	49.343	1.529	20.561	62.260	1.544
r20_10	507	0.102	7	19	*	(0.509)	(0.118)	*	3570.290
r20_11	558	0.113	8	(23)	*	(0.553)	(0.395)	(0.492)	(0.383)
r20_12	510	0.103	7	(21)	*	*	(0.483)	*	(0.284)
r20_13	535	0.108	6	16	*	*	(0.276)	(0.525)	3450.190
r20_14	514	0.104	7	(19)	*	*	(0.375)	*	(0.286)
r20_15	572	0.116	7	20	*	*	(0.153)	(0.591)	3012.200
r20_16	506	0.102	8	(23)	*	*	(0.562)	*	(0.321)
r20_17	513	0.104	6	22	(0.387)	(0.084)	31.653	(0.325)	7.988
r20_18	498	0.101	7	22	(0.377)	(0.470)	332.253	(0.453)	199.184
r20_19	552	0.112	7	20	*	*	1633.790	*	1841.170
r25_0	815	0.165	4	12	(0.369)	(0.234)	322.596	*	222.132
r25_1	760	0.154	4	∞	147.890	1.747	2.309	100.278	1.514
r25_2	759	0.153	5	12	*	*	(0.303)	*	3071.290
r25_3	811	0.164	5	12	*	*	(0.250)	*	3408.210
r25_4	760	0.154	5	(11)	*	*	(0.273)	*	(0.217)
r25_5	830	0.168	5	(11)	*	*	(0.325)	*	(0.245)
r25_6	775	0.157	5	(11)	*	*	(0.326)	*	(0.235)
r25_7	819	0.165	6	12	*	*	680.635	*	702.118
r25_8	747	0.151	4	12	*	*	747.049	*	1041.550
r25_9	778	0.157	5	13	*	*	(0.247)	*	2412.910
r25_10	737	0.149	5	12	*	*	2701.250	*	2366.970
r25_11	678	0.137	5	12	*	*	1877.590	(0.477)	1231.870
r25_12	842	0.170	5	(11)	*	*	(0.302)	*	(0.175)

Table 6: (continued)

Graph	m	ρ	s	γ^s_{club}	Time (or gap) Basic	DCF	NIF	ONI	All
r25_13	727	0.147	5	(12)	*	*	(0.315)	*	(0.211)
r25_14	859	0.174	5	(12)	*	*	(0.429)	(0.917)	(0.287)
r25_15	757	0.153	5	11	(0.429)	(0.429)	1005.170	*	1172.930
r25_16	767	0.155	4	14	2448.140	2073.940	145.488	2368.410	82.915
r25_17	779	0.157	4	13	(0.427)	(0.427)	567.317	(0.290)	772.974
r25_18	757	0.153	5	(13)	*	*	(0.256)	*	(0.270)
r25_19	773	0.156	5	11	*	*	857.871	*	2192.710
r30_0	1170	0.236	3	9	104.646	17.161	3.447	84.888	3.635
r30_1	1006	0.203	4	9	(0.930)	(0.930)	1237.110	(0.926)	1412.860
r30_2	1097	0.222	3	8	456.977	459.535	3.572	727.655	3.827
r30_3	1009	0.204	4	7	*	*	1039.640	(0.400)	341.652
r30_4	1055	0.213	3	8	75.333	19.594	8.424	70.872	4.103
r30_5	1013	0.205	3	8	10.640	6.053	2.356	36.380	2.699
r30_6	1151	0.233	3	8	540.266	538.472	8.517	158.779	34.024
r30_7	994	0.201	4	8	*	*	1349.180	*	1317.970
r30_8	1079	0.218	3	7	76.769	77.517	4.587	15.257	4.930
r30_9	986	0.199	4	9	(0.500)	(0.500)	239.526	*	9.844
r30_10	1070	0.216	3	8	969.584	399.802	11.903	81.043	13.447
r30_11	1021	0.206	4	8	(0.931)	(0.931)	727.344	*	274.798
r30_12	1004	0.203	3	9	113.273	36.115	21.793	194.722	4.149
r30_13	1121	0.226	4	9	(0.935)	(0.935)	3235.970	*	1419.850
r30_14	1067	0.216	4	7	*	*	1518.030	(0.933)	718.093
r30_15	1066	0.215	3	8	1413.860	1403.390	40.717	537.895	45.131
r30_16	985	0.199	3	9	22.855	247.451	2.668	280.710	3.089
r30_17	1171	0.237	3	8	2758.900	2754.500	63.259	28.361	73.259
r30_18	1042	0.211	3	7	304.251	306.965	1.997	781.274	3.261
r30_19	1083	0.219	3	10	31.169	4.212	3.760	27.893	2.870
r35_0	1287	0.260	2	6	3.338	2.169	1.420	3.120	1.919
r35_1	1298	0.262	2	7	4.165	3.447	1.514	3.541	2.200
r35_2	1375	0.278	3	6	1722.250	1739.530	44.913	2724.990	49.063
r35_3	1399	0.283	2	6	4.633	2.683	1.575	3.853	2.138
r35_4	1307	0.264	3	6	609.173	604.102	4.197	2940.610	4.399
r35_5	1479	0.299	3	6	2504.140	2485.720	41.060	2308.270	36.255
r35_6	1402	0.283	3	6	2689.000	2685.540	43.119	(0.167)	58.469
r35_7	1394	0.282	2	6	4.914	5.679	1.576	3.105	2.309
r35_8	1338	0.270	3	6	2649.800	2639.620	23.401	473.232	40.904
r35_9	1454	0.294	2	0	3.230	1.997	1.623	3.338	1.950
r35_10	1367	0.276	3	6	(0.167)	(0.167)	115.395	23.088	108.109
r35_11	1488	0.301	2	6	4.649	2.777	1.685	3.478	2.153
r35_12	1210	0.244	3	7	2974.020	2978.080	3.994	420.566	21.825
r35_13	1414	0.286	3	6	(0.410)	(0.410)	294.376	0.000	246.374
r35_14	1195	0.241	3	6	1562.770	1549.710	11.872	3068.000	78.297
r35_15	1440	0.291	2	7	4.259	2.589	1.654	3.276	2.106
r35_16	1310	0.265	2	6	3.323	3.198	1.498	3.526	1.981
r35_17	1394	0.282	3	6	(0.267)	(0.267)	43.197	1512.670	9.548
r35_18	1325	0.268	3	6	2816.150	2797.600	50.358	2363.060	11.030
r35_19	1251	0.253	3	6	937.900	938.680	2.621	15.600	3.073
r40_0	1623	0.328	2	5	5.070	5.897	1.841	8.315	2.824
r40_1	1634	0.330	2	5	5.257	6.146	1.778	8.034	2.418
r40_2	1616	0.326	2	5	5.928	6.833	1.826	5.554	2.621

Table 6: (continued)

Graph	m	ρ	s	γ^s_{club}	Time (or gap) Basic	DCF	NIF	ONI	All
r40_3	1678	0.339	2	5	5.818	6.801	2.106	6.817	3.120
r40_4	1729	0.349	2	5	5.492	6.428	1.654	6.100	2.699
r40_5	1780	0.360	2	4	4.680	5.663	1.732	4.556	2.683
r40_6	1727	0.349	2	5	5.990	6.910	2.200	5.741	3.073
r40_7	1705	0.344	2	5	5.710	6.614	2.059	7.738	3.057
r40_8	1592	0.322	2	5	5.444	6.302	1.872	6.068	2.559
r40_9	1961	0.396	2	5	6.630	7.613	2.684	7.597	3.806
r40_10	1774	0.358	2	5	5.725	6.755	2.059	6.162	2.777
r40_11	1674	0.338	2	5	5.008	5.975	2.012	5.336	2.683
r40_12	1744	0.352	2	5	5.944	6.786	2.137	5.335	2.933
r40_13	1795	0.363	2	5	6.287	7.207	2.527	5.990	3.401
r40_14	1511	0.305	2	5	5.522	6.505	1.966	5.070	2.902
r40_15	1627	0.329	2	5	5.523	6.412	1.997	6.942	2.995
r40_16	1769	0.357	2	5	5.913	6.817	2.184	5.694	3.042
r40_17	1652	0.334	2	5	4.040	3.541	1.622	4.712	2.340
r40_18	1740	0.352	2	5	6.739	7.675	2.153	8.767	2.917
r40_19	1716	0.347	2	5	11.201	11.872	1.857	7.098	2.668
r45_0	2270	0.459	2	4	8.892	10.093	1.310	4.914	2.309
r45_1	2345	0.474	1	4	1.669	2.714	1.482	1.654	2.605
r45_2	2211	0.447	1	3	1.607	2.714	1.435	1.591	2.558
r45_3	2016	0.407	2	4	7.082	8.236	2.059	27.878	2.948
r45_4	1951	0.394	2	4	5.226	6.147	2.059	5.039	3.042
r45_5	1902	0.384	2	4	3.619	4.602	0.889	2.636	1.856
r45_6	2104	0.425	2	4	10.078	11.138	2.606	7.316	3.338
r45_7	1963	0.397	2	4	4.415	5.413	1.825	5.164	2.761
r45_8	2099	0.424	1	5	1.560	2.496	1.388	1.544	2.324
r45_9	1755	0.355	2	5	9.500	10.452	1.966	17.815	2.730
r45_10	2146	0.434	1	5	1.654	2.527	1.435	1.607	2.464
r45_11	2167	0.438	2	3	4.508	5.616	2.060	4.493	3.120
r45_12	1941	0.392	2	3	3.775	4.711	1.732	3.760	2.730
r45_13	2065	0.417	2	4	5.803	6.926	2.059	6.443	3.120
r45_14	2006	0.405	2	4	4.680	5.725	2.200	4.742	2.917
r45_15	1892	0.382	2	4	6.115	7.176	1.778	5.242	2.714
r45_16	2047	0.414	2	5	7.784	8.861	2.106	9.064	3.089
r45_17	2005	0.405	2	4	5.116	6.162	2.184	6.334	3.042
r45_18	2134	0.431	1	4	1.654	2.418	1.435	1.591	2.356
r45_19	1960	0.396	2	4	4.197	5.194	1.014	3.713	1.856
r50_0	2377	0.480	1	3	1.685	2.871	1.513	1.669	2.668
r50_1	2558	0.517	1	3	1.747	3.058	1.576	1.716	2.855
r50_2	2620	0.529	1	3	1.810	3.135	1.606	1.794	2.902
r50_3	2581	0.521	1	3	1.731	3.026	1.560	1.700	2.855
r50_4	2309	0.466	1	3	1.638	2.808	1.482	1.576	2.590
r50_5	2339	0.473	1	3	1.653	2.854	1.529	1.669	2.698
r50_6	2376	0.480	1	3	1.716	2.917	1.513	1.700	2.730
r50_7	2437	0.492	1	3	1.654	2.902	1.544	1.622	2.730
r50_8	2346	0.474	1	4	1.763	2.683	1.466	1.669	2.590
r50_9	2367	0.478	1	4	1.669	2.699	1.513	1.654	2.652
r50_10	2355	0.476	1	3	1.669	2.901	1.497	1.669	2.667
r50_11	2459	0.497	1	3	1.732	2.964	1.576	1.700	2.746
r50_12	2269	0.458	1	3	1.638	2.792	1.482	1.622	2.558

Table 6: (continued)

Graph	m	ρ	s	γ^s_{club}	Time (or gap)				
					Basic	DCF	NIF	ONI	All
r50_13	2466	0.498	1	3	1.732	2.980	1.544	1.732	2.761
r50_14	2431	0.491	1	3	1.701	2.933	1.576	1.700	2.745
r50_15	2332	0.471	1	4	1.700	2.901	1.498	1.654	2.668
r50_16	2285	0.462	1	4	1.669	2.777	1.482	1.622	2.589
r50_17	2195	0.443	1	3	1.560	2.714	1.420	1.545	2.496
r50_18	2248	0.454	1	3	1.638	2.777	1.451	1.576	2.590
r50_19	2300	0.465	1	3	1.607	2.808	1.482	1.622	2.668